Stochastic Signals and Systems

Contents

2 Stochastic Processes

Stochastic (random) processes arise as a result of the following situations:

a) The system that generates the process may inherently possess random elements, e.g. the emission of particles in radioactive materials.

b) The system may be basically deterministic but of such a complexity that it is impossible to model it without probabilistic means.

c) Even if the simplicity of the system allows a complete deterministic description, the data obtained by observing the system are contaminated by measurement errors.

2.1 Fundamentals

2.1.1 Definition of Stochastic Processes

A stochastic process $(X_t)_{t\in\mathcal{T}}$ is a family of random variables, indexed by t, where t belongs to some given index set $\mathcal{T}$.

If t takes a continuous domain of real values (finite or infinite), $(X_t)_{t\in\mathcal{T}}$ is said to be a continuous time stochastic process.

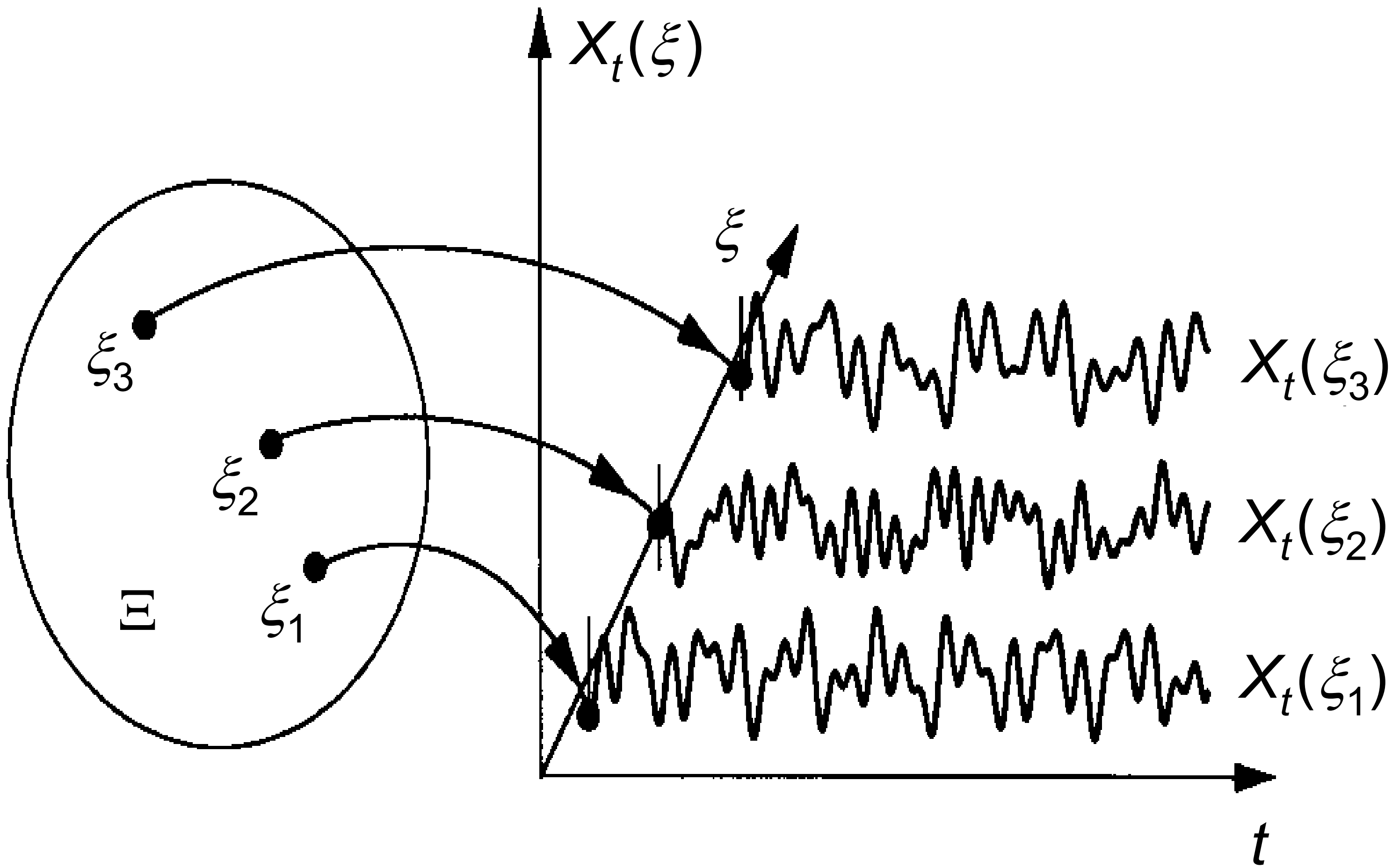
If t takes a discrete set of values (typically $t = \ldots -2,-1,0, 1,2,\ldots$) then $(X_t)_{t\in\mathcal{T}}$ is said to be a discrete time stochastic process or stochastic sequence.

2.1.2 Sample Function and Ensembles

An observed record of a stochastic process is merely one record out of the whole collection of possible records.

The collection of all possible records is called ensemble and each particular record is called sample function or realization of the stochastic process.

Thus, we can interpret the sample space Ξ to consist of a set of elementary events ξ, where each corresponds to a particular sample function such that we can denote the various sample functions by $X_t(\xi_1), X_t(\xi_2), \ldots$


$X_t(\xi)$
ξ
ξ_3
ξ_2
ξ_1
Ξ
$X_t(\xi_3)$
$X_t(\xi_2)$
$X_t(\xi_1)$
t

2.1.3 Probabilistic Description of Stochastic Processes

Generally, we may wish to investigate the behaviour of a stochastic process over all time points, e.g. if we want to determine the probability that a stochastic process remains within certain limits, namely

$$P(a \leq X_t \leq b, \forall t \in \mathcal{T}).$$

Thus, to describe the properties of the complete process, it seems to be necessary to consider an infinite dimensional probability distribution.

Fortunately, it turns out that under fairly general conditions the probabilistic properties of the complete process can be specified by its behaviour at finite numbers of time points.

Theorem:

For any positive integer n, let $t_1,\ldots,t_n$ be any admissible set of values of t. Then under general conditions the probabilistic structure of the stochastic process is completely defined if we are given the joint probability distribution of $X_{t_1},\ldots,X_{t_n}$ for all values of n and for all choices of $t_1,\ldots,t_n$.

We do not attempt to state the general conditions under which the above result holds.

However, for all practical purposes it seems to be intuitively reasonable that the joint distribution of $X_{t_1},\ldots,X_{t_n}$ for an arbitrarily large but finite number n of time points suffice to describe the stochastic process.

The joint probability distribution of $X_{t_1},\ldots,X_{t_n}$ is denoted by

$$F_{\mathbf{X}}(x_1,\ldots,x_n;t_1,\ldots,t_n) = P(X_{t_1} \le x_1,\ldots,X_{t_n} \le x_n)$$

and the corresponding density function by

$$f_{\mathbf{X}}(x_1,\ldots,x_n;t_1,\ldots,t_n) = \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{\mathbf{X}}(x_1,\ldots,x_n;t_1,\ldots,t_n).$$

2.1.4 Complex Stochastic Processes

So far only real valued stochastic processes have been considered.

Nevertheless, in many applications it is more convenient to regard them as complex valued stochastic processes, e.g. if a quadrature demodulation or Hilbert transform is involved.

Let $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ be real valued stochastic processes. Then the process $(Z_t)_{t\in\mathcal{T}}$ formed by

$$Z_t = X_t + jY_t$$

is called complex stochastic process.

The probabilistic structure of $(Z_t)_{t\in\mathcal{T}}$ is specified by the joint distribution of the sets of random variables

$$\left\{ X_{t_1},\ldots,X_{t_n},Y_{t_1},\ldots,Y_{t_n} \right\},$$

whose joint probability distribution and density function can be expressed by

$$F_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1,\ldots,t_n) = P(X_{t_1} \le x_1,\ldots,X_{t_n} \le x_n,Y_{t_1} \le y_1,\ldots,Y_{t_n} \le y_n)$$

resp.

$$f_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1,\ldots,t_n) = \frac{\partial^{2n}}{\partial x_1\cdots\partial x_n\partial y_1\cdots\partial y_n}F_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1,\ldots,t_n).$$

2.1.5 Moment Functions

For real valued stochastic processes we define:

a) Mean Function

$$\mu_X(t) = \mathrm{E}(X_t) = \int_{-\infty}^{\infty} x\, f_X(x;t)\,dx.$$

b) Second Order Moment Function

$$r_{XX}(t_1,t_2) = \mathrm{E}\left(X_{t_1} X_{t_2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1\, x_2\, f_{\mathbf{X}}(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

with $r_{XX}(t_1,t_2) = r_{XX}(t_2,t_1).$

c) Covariance Function

$$c_{XX}(t_1,t_2) = \mathrm{Cov}\left(X_{t_1}, X_{t_2}\right) = \mathrm{E}\left(\left(X_{t_1} - \mu_X(t_1)\right)\left(X_{t_2} - \mu_X(t_2)\right)\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x_1 - \mu_X(t_1)\right)\left(x_2 - \mu_X(t_2)\right) f_{\mathbf{X}}(x_1, x_2; t_1, t_2)\, dx_1\, dx_2$$

with $c_{XX}(t_1,t_2) = c_{XX}(t_2,t_1)$.

Employing the mean and second order moment function the covariance function can be expressed by

$$c_{XX}(t_1,t_2) = \mathrm{E}\left(X_{t_1} X_{t_2}\right) - \mu_X(t_1)\mu_X(t_2)$$

$$= r_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X(t_2).$$

d) Variance Function

$$\sigma_X^2(t) = \mathrm{Var}(X_t) = c_{XX}(t,t) = \mathrm{E}\left(\left(X_t - \mu_X(t)\right)^2\right)$$

$$= \int_{-\infty}^{\infty} \left(x - \mu_X(t)\right)^2 f_X(x;t)\,dx.$$

In terms of the mean and the second order moment function the variance function can be determined by

$$\sigma_X^2(t) = \mathrm{E}\left(X_t^2\right) - \mu_X^2(t) = r_{XX}(t,t) - \mu_X^2(t).$$

e) Correlation Function

$$\rho_{XX}(t_1,t_2) = \frac{c_{XX}(t_1,t_2)}{\sqrt{\sigma_X^2(t_1)\sigma_X^2(t_2)}} \quad \text{with} \quad \rho_{XX}(t_1,t_2) = \rho_{XX}(t_2,t_1).$$

The above definition follows the definition in the statistical literature, which is a straight forward generalization of the definition of the correlation coefficient of random variables.

However, in the engineering literature unfortunately the the second order moment function is often called correlation function.

f) Cross Second Order Moment Function

$$r_{XY}(t_1,t_2) = \mathrm{E}\left(X_{t_1} Y_{t_2}\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x y \, f_{XY}(x,y;t_1,t_2)\,dx\,dy$$

with $r_{XY}(t_1,t_2) = r_{YX}(t_2,t_1)$.

g) Cross Covariance Function

$$c_{XY}(t_1,t_2) = \mathrm{Cov}\left(X_{t_1}, Y_{t_2}\right) = \mathrm{E}\left(\left(X_{t_1} - \mu_X(t_1)\right)\left(Y_{t_2} - \mu_Y(t_2)\right)\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x - \mu_X(t_1)\right)\left(y - \mu_Y(t_2)\right) f_{XY}(x,y;t_1,t_2)\,dx\,dy$$

with $c_{XY}(t_1,t_2) = c_{YX}(t_2,t_1)$.

By utilising the mean functions and the cross second order moment function we can derive

$$c_{XY}(t_1,t_2) = \mathrm{E}\left(X_{t_1} Y_{t_2}\right) - \mu_X(t_1)\mu_Y(t_2)$$

$$= r_{XY}(t_1,t_2) - \mu_X(t_1)\mu_Y(t_2).$$

For complex valued stochastic processes we define:

a) Mean Function

$$\mu_Z(t) = \mathrm{E}(Z_t) = \mathrm{E}(X_t) + j\,\mathrm{E}(Y_t)$$

$$= \int_{-\infty}^{\infty} x\, f_X(x;t)\,dx + j \int_{-\infty}^{\infty} y\, f_Y(y;t)\,dy = \mu_X(t) + j\mu_Y(t).$$

b) Second Order Moment Function

$$r_{ZZ}(t_1,t_2) = \mathrm{E}\left(Z_{t_1} Z_{t_2}^*\right) = \mathrm{E}\Big((X_{t_1} + jY_{t_1})(X_{t_2} - jY_{t_2})\Big)$$

$$= r_{XX}(t_1,t_2) + r_{YY}(t_1,t_2) + j\left\{r_{YX}(t_1,t_2) - r_{XY}(t_1,t_2)\right\}$$

with $r_{ZZ}(t_1,t_2) = r_{ZZ}(t_2,t_1)^*$.

c) Covariance Function

$$c_{ZZ}(t_1,t_2)=c_{XX}(t_1,t_2)+c_{YY}(t_1,t_2)+j\left\{c_{YX}(t_1,t_2)-c_{XY}(t_1,t_2)\right\}$$

with $c_{ZZ}(t_1,t_2)=c_{ZZ}(t_2,t_1)^*$.

d) Variance Function

$$\sigma_Z^2(t)=c_{ZZ}(t,t)=c_{XX}(t,t)+c_{YY}(t,t)=\sigma_X^2(t)+\sigma_Y^2(t)$$

as $c_{YX}(t_1,t_2)=c_{XY}(t_2,t_1) \Rightarrow c_{YX}(t,t)=c_{XY}(t,t)$.

e) Correlation Function

$$\rho_{ZZ}(t_1,t_2)=\frac{c_{ZZ}(t_1,t_2)}{\sqrt{\sigma_Z^2(t_1)\sigma_Z^2(t_2)}}$$ with $\rho_{ZZ}(t_1,t_2)=\rho_{ZZ}(t_2,t_1)^*$.

f) Cross Second Order Moment Function

$$r_{ZW}(t_1,t_2) = \mathrm{E}\left(Z_{t_1} W_{t_2}^*\right) = \mathrm{E}\left((X_{t_1} + jY_{t_1})(U_{t_2} - jV_{t_2})\right)$$

$$= r_{XU}(t_1,t_2) + r_{YV}(t_1,t_2) + j\left\{r_{YU}(t_1,t_2) - r_{XV}(t_1,t_2)\right\}$$

with $r_{ZW}(t_1,t_2) = r_{WZ}(t_2,t_1)^*$.

g) Cross Covariance Function

$$c_{ZW}(t_1,t_2) = c_{XU}(t_1,t_2) + c_{YV}(t_1,t_2) + j\left\{c_{YU}(t_1,t_2) - c_{XV}(t_1,t_2)\right\}$$

with $c_{ZW}(t_1,t_2) = c_{WZ}(t_2,t_1)^*$.

2.2 Some Particular Processes

2.2.1 Poisson Process

A Poisson process is useful for modelling the random time points of the occurrence of events.

Let $N(t_1,t_2)$ be the model for counting the number of events occurring in the time interval $(t_1,t_2]$. Then one can show that $N(t_1,t_2)$ exhibits the following properties.

- $N(t_1,t_2)$ is a Poisson distributed random variable with parameter $\lambda(t_2-t_1)>0$, i.e.

$$P\left(N(t_1,t_2)=k\right)=e^{-\lambda(t_2-t_1)}\frac{\left(\lambda(t_2-t_1)\right)^k}{k!}.$$

- $N(t_1,t_2)$ and $N(t_3,t_4)$ are independent if the intervals $(t_1,t_2]$ and $(t_3,t_4]$ are disjoint.

The random process

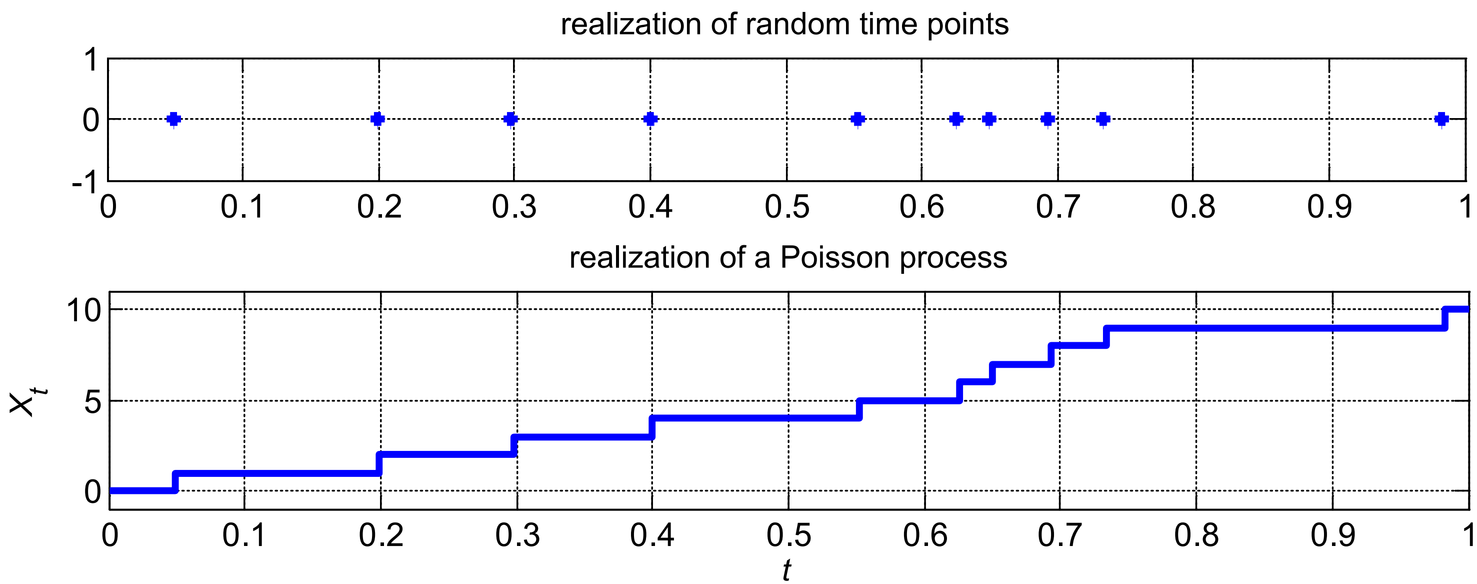
$$X_t = N(0,t)$$

that counts the number of events in the interval $(0,t]$ is called Poisson process.

Since any two increments of the form

$$X_{t_2} - X_{t_1} = N(t_1,t_2) \quad \text{and} \quad X_{t_4} - X_{t_3} = N(t_3,t_4)$$

are independent if $t_1 < t_2 \leq t_3 < t_4$, a Poisson process is said to possess independent increments.



Mean, variance and covariance of a Poisson process are given by

$$\mu_X(t) = \mathrm{E}(X_t) = \lambda t,$$

$$\sigma_X^2(t) = \mathrm{Var}(X_t) = \lambda t \;\text{ and }\; c_{XX}(t_1,t_2) = \lambda \cdot \min(t_1,t_2).$$

2.2.2 Random Walk

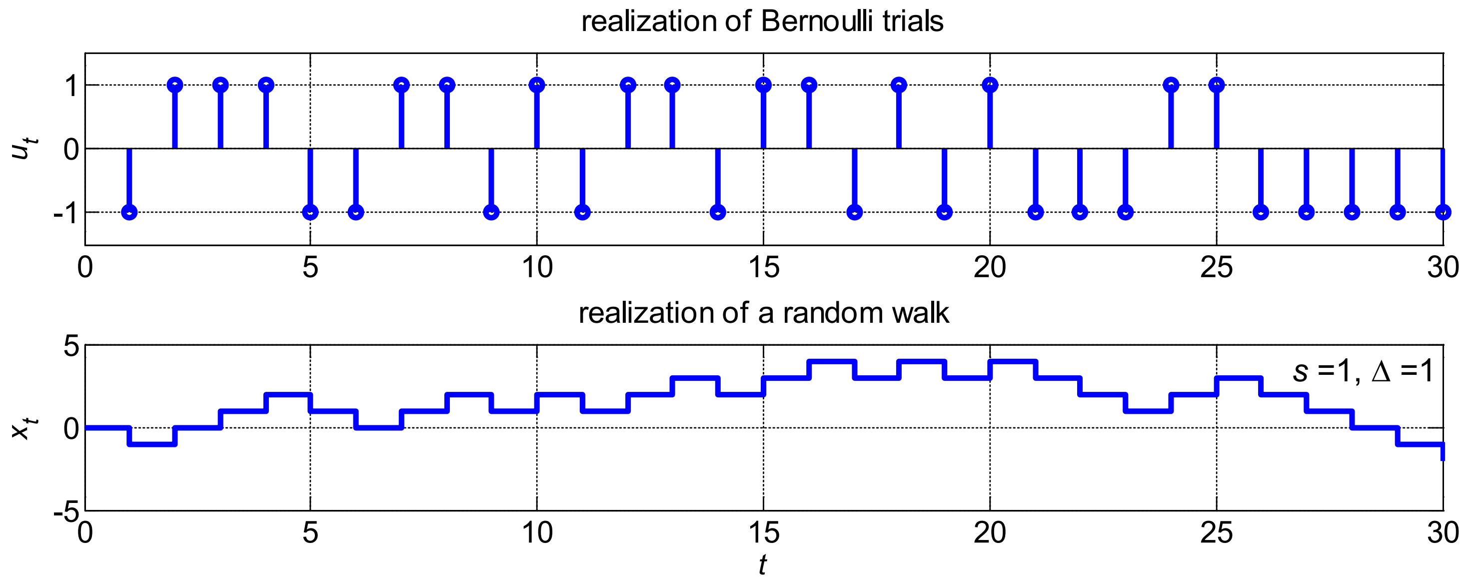
Let U_n for $n = 1,2,\dots$ be random variables describing a sequence of independent and identically distributed trails (Bernoulli) with

$$P(U_n = u) = \begin{cases} p & u = 1 \\ 1-p & u = -1 \end{cases} \qquad n = 1,2,\dots .$$

Furthermore, we define the random process

$$X_t = \sum_{n=1}^{m} s\,U_n \quad \text{with} \quad m = \lfloor t/\Delta \rfloor,$$

where s and Δ denote the step height and step width respectively. In the case of $p = 0.5$ the random process X_t is termed a random walk.



Mean and variance of a random walk are given by

$$\mu_X(t) = s\sum_{n=1}^{m} \mathrm{E}(U_n) = s\,m \cdot (2p-1) = 0,$$

$$\sigma_X^2(t) = s^2\sum_{n=1}^{m} \mathrm{Var}(U_n) = s^2\,m.$$

Relying on the central limit theorem one can assert that $X_t \sim \mathcal{N}(0, s^2 m)$ approximately holds for large $m = \lfloor t/\Delta \rfloor$.

2.2.4 Wiener Process (Brownian motion)

We now examine the limiting form of a random walk X_t as $\Delta \to 0$. The variance function of X_t is known to be

$$\sigma_X^2(t) = s^2 m \quad \text{with} \quad m = \lfloor t/\Delta \rfloor.$$

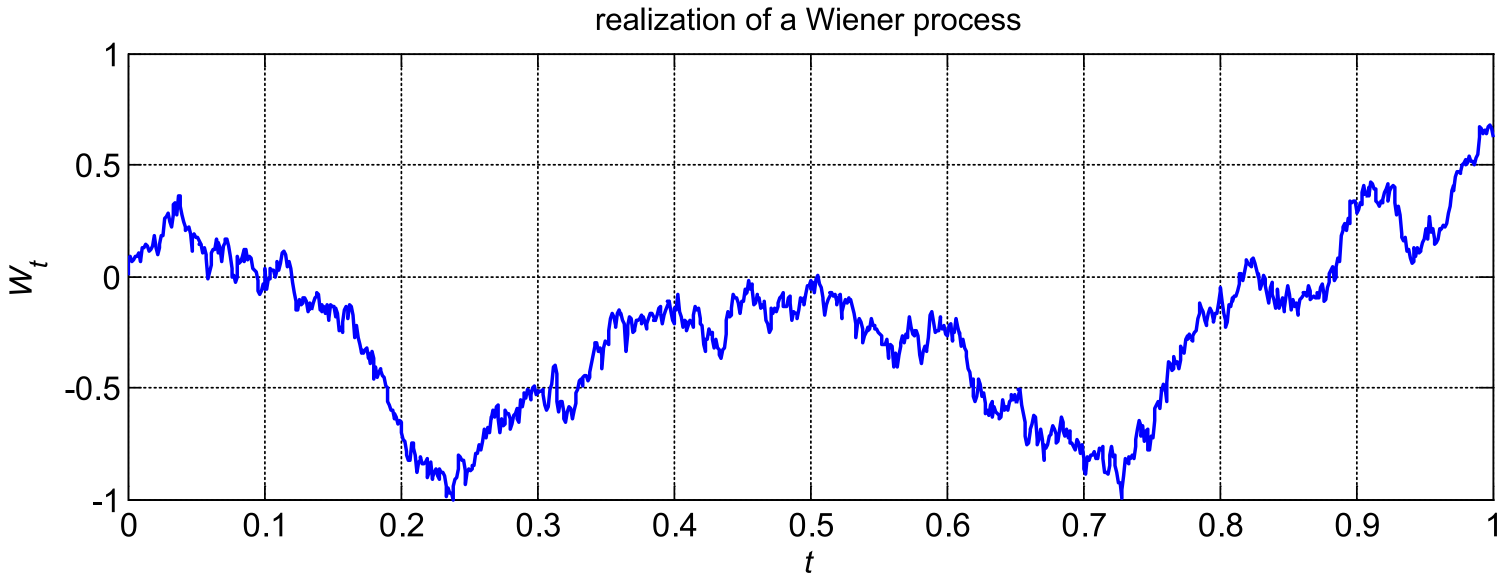
To obtain a meaningful result as $\Delta \to 0$ the step height s should be proportional to $\sqrt{\Delta}$, i.e.

$$s^2 = \alpha \Delta.$$

The limit of X_t as $\Delta \to 0$ (almost surely) is then the continuous process

$$W_t = \lim_{\Delta \to 0} X_t \,.$$

This random process is called Wiener process. It is often used as a model for Brownian motion.



Mean, variance and covariance of a Wiener process are given by

$$\mu_W(t) = \mathrm{E}(W_t) = 0,$$

$$\sigma_W^2(t) = \mathrm{Var}(W_t) = \alpha t \quad \text{and} \quad c_{WW}(t_1,t_2) = \alpha \cdot \min(t_1,t_2).$$

Furthermore, exploiting a suitable central limit theorem one can prove that $W_t \sim \mathcal{N}(0,\alpha t)$.

Thus, a Wiener process is a correlated random process whose mean is zero, variance increases linearly with time and probability density function is Gaussian.

2.2.4 Markov Process

A stochastic process X_t is called Markov process if

$$P(X_{t_n} \leq x_n | X_{t_1} = x_1, \ldots, X_{t_{n-1}} = x_{n-1}) = P(X_{t_n} \leq x_n | X_{t_{n-1}} = x_{n-1})$$

holds for $t_1 < t_2 < \ldots < t_{n-1} < t_n$. That is, the past $t_1, t_2, \ldots, t_{n-2}$ has no influence on the statistical properties of the future t_n if the present t_{n-1} is specified.

Example:

A stochastic process with independent increments and $X_{t=0} = 0$ represents a Markov process.

Thus, the Poisson process and the Wiener process are examples for a Markov process.

2.2.5 Gauss Process

A stochastic process is called Gauss process if, for any admissible $t_1,\dots,t_n$ the $X_{t_1},\dots,X_{t_n}$ possess a multivariate Gaussian probability density function, i.e.

$$\mathbf{X} = (X_{t_1},\dots,X_{t_n})^T \sim \mathcal{N}_n(\boldsymbol{\mu}_X, \mathbf{C}_{XX})$$

with

$$\boldsymbol{\mu}_X = \begin{pmatrix} \mathrm{E}(X_{t_1}) \\ \mathrm{E}(X_{t_2}) \\ \vdots \\ \mathrm{E}(X_{t_n}) \end{pmatrix} = \begin{pmatrix} \mu_X(t_1) \\ \mu_X(t_2) \\ \vdots \\ \mu_X(t_n) \end{pmatrix}$$

and

$$\mathbf{C}_{XX} = \begin{pmatrix} c_{XX}(t_1,t_1) & c_{XX}(t_1,t_2) & \cdots & c_{XX}(t_1,t_n) \\ c_{XX}(t_2,t_1) & c_{XX}(t_2,t_2) & \cdots & c_{XX}(t_2,t_n) \\ \vdots & \vdots & \ddots & \vdots \\ c_{XX}(t_n,t_1) & c_{XX}(t_n,t_2) & \cdots & c_{XX}(t_n,t_n) \end{pmatrix},$$

where $c_{XX}(t_n,t_m) = \mathrm{E}\Big(\big(X_{t_n} - \mu_X(t_n)\big)\big(X_{t_m} - \mu_X(t_m)\big)\Big).$

Hence, a Gauss process is completely described by its mean function $\mu_X(t)$ and covariance function $c_{XX}(t_1,t_2)$.

Example:

The Wiener process is a Gauss process.

2.3 Stationary Processes

2.3.1 Real Valued Stationary Processes

A real valued stochastic process $(X_t)_{t\in\mathcal{T}}$ is called strict-sense stationary if, for any admissible $t_1,\ldots,t_n$ and any τ, the joint probability distribution of

$$X_{t_1},\ldots,X_{t_n}$$

is identical with the joint probability distribution of

$$X_{t_1+\tau},\ldots,X_{t_n+\tau}\,.$$

Consequently, the distribution does not depend on τ, i.e.

$$F_{\mathbf{X}}(x_1,\ldots,x_n;t_1,\ldots,t_n)=F_{\mathbf{X}}(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau)$$

and, if the density function exists

$$f_{\mathbf{X}}(x_1,\ldots,x_n;t_1,\ldots,t_n) = f_{\mathbf{X}}(x_1,\ldots,x_n;t_1+\tau,\ldots,t_n+\tau).$$

Implications:

1) The univariate density function is independent of *t*.

$$f_X(x;t) = f_X(x;t+\tau) = f_X(x) \;\Rightarrow\; \mu_X(t) = \mu_X = \text{const.}$$

2) The bivariate density function depends only on the time difference $t_1 - t_2$.

$$\begin{aligned} f_{\mathbf{X}}(x_1,x_2;t_1,t_2) &= f_{\mathbf{X}}(x_1,x_2;t_1-t_2,0) = f_{\mathbf{X}}(x_1,x_2;t_1-t_2) \\ &= f_{\mathbf{X}}(x_1,x_2;0,t_2-t_1) = f_{\mathbf{X}}(x_1,x_2;t_2-t_1) \end{aligned}$$

$$\Rightarrow\; c_{XX}(t_1,t_2) = c_{XX}(t_1-t_2) = c_{XX}(t_2-t_1).$$

A real valued stochastic process $(X_t)_{t\in\mathcal{T}}$ is said to be stationary up to order m if, for any admissible $t_1,\ldots,t_n$ and any τ, the joint moments up to order m of

$$X_{t_1},\ldots,X_{t_n}$$

exist and equal the corresponding joint moments of

$$X_{t_1+\tau},\ldots,X_{t_n+\tau}\,.$$

Thus,

$$\mathrm{E}\left(X_{t_1}^{m_1}\cdots X_{t_n}^{m_n}\right)=\mathrm{E}\left(X_{t_1+\tau}^{m_1}\cdots X_{t_n+\tau}^{m_n}\right)$$

for any τ, and all positive integers $m_1,\ldots,m_n$ satisfying

$$m_1+\ldots+m_n\le m.$$

A real valued stochastic process $(X_t)_{t\in\mathcal{T}}$ that is stationary up to order $m = 2$ is called wide-sense stationary.

Let $(X_t)_{t\in\mathcal{T}}$ be a real valued wide-sense stationary stochastic process then we have

1) $\mathrm{E}(X_t) = \mu_X$, a constant independent of t, i.e. the same mean value at all time points,
2) $\mathrm{Var}(X_t) = \mathrm{E}(X_t^2) - \mu_X^2 = \sigma_X^2$, a constant independent of t, i.e. the same variance at all time points,
3) $c_{XX}(t_1,t_2) = c_{XX}(t_1 - t_2) = c_{XX}(t_2 - t_1)$, i.e. the covariance depends only on the interval between the time points.

Exercise 2.3-1:
Single tone stochastic process

2.3.2 Complex Valued Stationary Processes

A complex valued stochastic process $(Z_t = X_t + jY_t)_{t\in\mathcal{T}}$ is called strict-sense stationary if, for any admissible $t_1,\ldots,t_n$ and any τ, the joint probability distributions of

$$X_{t_1},\ldots,X_{t_n},Y_{t_1},\ldots,Y_{t_n} \quad \text{and} \quad X_{t_1+\tau},\ldots,X_{t_n+\tau},Y_{t_1+\tau},\ldots,Y_{t_n+\tau}$$

are identical.

Thus, the distribution function does not depend on τ, i.e.

$$F_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1,\ldots,t_n) = F_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1+\tau,\ldots,t_n+\tau).$$

Furthermore, if the density function exists, we can write

$$f_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1,\ldots,t_n) = f_{\mathbf{XY}}(x_1,\ldots,x_n,y_1,\ldots,y_n;t_1+\tau,\ldots,t_n+\tau).$$

Implications:

1) The univariate density functions are independent of *t*.

$$\left.\begin{aligned} f_X(x;t) &= f_X(x;t+\tau) = f_X(x) \\ f_Y(y;t) &= f_Y(y;t+\tau) = f_Y(y) \end{aligned}\right\} \Rightarrow \mu_Z(t) = \mu_Z = \mu_X + j\mu_Y = \text{const.}$$

2) The bivariate density functions depend only on the time difference $t_1 - t_2$.

$$\left.\begin{aligned} f_{\mathbf{X}}(x_1,x_2;t_1,t_2) &= f_{\mathbf{X}}(x_1,x_2;t_1-t_2) = f_{\mathbf{X}}(x_1,x_2;t_2-t_1) \\ f_{\mathbf{Y}}(y_1,y_2;t_1,t_2) &= f_{\mathbf{Y}}(y_1,y_2;t_1-t_2) = f_{\mathbf{Y}}(y_1,y_2;t_2-t_1) \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow c_{XX}(t_1,t_2) = c_{XX}(t_1-t_2) = c_{XX}(t_2-t_1)$$
$$c_{YY}(t_1,t_2) = c_{YY}(t_1-t_2) = c_{YY}(t_2-t_1)$$

and

$$f_{XY}(x,y;t_1,t_2) = f_{XY}(x,y;t_1-t_2)$$
$$= f_{YX}(y,x;t_2-t_1) = f_{YX}(y,x;t_2,t_1)$$
$$\Rightarrow c_{XY}(t_1,t_2) = c_{XY}(t_1-t_2) = c_{YX}(t_2-t_1) = c_{YX}(t_2,t_1).$$

Consequently,

$$c_{ZZ}(t_1,t_2) = c_{XX}(t_1-t_2) + c_{YY}(t_1-t_2) +$$
$$+j\{c_{YX}(t_1-t_2) - c_{XY}(t_1-t_2)\} = c_{ZZ}(t_1-t_2)$$

with $c_{ZZ}(t_1,t_2) = c_{ZZ}(t_1-t_2) = c_{ZZ}(t_2-t_1)^* = c_{ZZ}(t_2,t_1)^*.$

2.3.3 Moment Functions for Stationary Processes

For real valued (wide-sense) stationary stochastic processes we can derive:

a) *Mean Function*

$$\mu_X = \mathrm{E}(X_t) = const.$$

b) *Second Order Moment Function*

$$r_{XX}(\tau) = \mathrm{E}\left(X_{t+\tau} X_t\right) \text{ with } r_{XX}(\tau) = r_{XX}(-\tau).$$

c) *Covariance Function*

$$c_{XX}(\tau) = \mathrm{E}\left((X_{t+\tau} - \mu_X)(X_t - \mu_X)\right) = \mathrm{E}\left(X_{t+\tau} X_t\right) - \mu_X^2$$

$$= r_{XX}(\tau) - \mu_X^2 \text{ with } c_{XX}(\tau) = c_{XX}(-\tau).$$

d) *Variance Function*

$$\sigma_X^2 = \mathrm{Var}(X_t) = c_{XX}(0) = r_{XX}(0) - \mu_X^2 = const.$$

e) *Correlation Function*

$$\rho_{XX}(\tau) = \frac{c_{XX}(\tau)}{\sigma_X^2} = \frac{c_{XX}(\tau)}{c_{XX}(0)} \text{ with } \rho_{XX}(\tau) = \rho_{XX}(-\tau).$$

f) *Cross Second Order Moment Function*

$$r_{XY}(\tau) = \mathrm{E}\left(X_{t+\tau} Y_t\right) \text{ with } r_{XY}(\tau) = r_{YX}(-\tau).$$

g) *Cross Covariance Function*

$$c_{XY}(\tau) = \mathrm{E}\left((X_{t+\tau} - \mu_X)(Y_t - \mu_Y)\right) = \mathrm{E}\left(X_{t+\tau} Y_t\right) - \mu_X \mu_Y$$

$$= r_{XY}(\tau) - \mu_X \mu_Y \text{ with } c_{XY}(\tau) = c_{YX}(-\tau).$$

Properties of the Moment Functions

(1) $\left|r_{XX}(\tau)\right| \leq r_{XX}(0),\ \left|c_{XX}(\tau)\right| \leq c_{XX}(0),\ \left|\rho_{XX}(\tau)\right| \leq 1.$

(2) $r_{XX}(\tau), c_{XX}(\tau), \rho_{XX}(\tau)$
are so-called positive semi-definite functions.

(3) $\left|r_{XY}(\tau)\right| \leq \sqrt{r_{XX}(0)\cdot r_{YY}(0)} \leq \frac{1}{2}\left(r_{XX}(0)+r_{YY}(0)\right),$

$\left|c_{XY}(\tau)\right| \leq \sqrt{c_{XX}(0)\cdot c_{YY}(0)} \leq \frac{1}{2}\left(c_{XX}(0)+c_{YY}(0)\right).$

Exercise 2.3-2:
Proof of the properties 1) – 3)

Remarks:

Two stationary real valued stochastic processes $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ are said to be

- uncorrelated, if for all τ holds

  $$c_{XY}(\tau) = \mathrm{E}\big((X_{t+\tau} - \mu_X)(Y_t - \mu_Y)\big) = \mathrm{E}\big(X_{t+\tau} Y_t\big) - \mu_X \mu_Y = 0$$

  or equivalently

  $$r_{XY}(\tau) = \mathrm{E}\big(X_{t+\tau} Y_t\big) = \mathrm{E}\big(X_{t+\tau}\big)\mathrm{E}\big(Y_t\big) = \mu_X \mu_Y,$$

- orthogonal, if for all τ holds

  $$r_{XY}(\tau) = \mathrm{E}\big(X_{t+\tau} Y_t\big) = 0.$$

For complex valued (wide-sense) stationary stochastic processes we can deduce:

a) Mean Function

$$\mu_Z = \mathrm{E}(Z_t) = \mathrm{E}(X_t) + j\,\mathrm{E}(Y_t) = \mu_X + j\mu_Y = const.$$

b) Second Order Moment Function

$$r_{ZZ}(\tau) = \mathrm{E}\left(Z_{t+\tau} Z_t^*\right) = \mathrm{E}\left((X_{t+\tau} + jY_{t+\tau})(X_t - jY_t)\right)$$

$$= r_{XX}(\tau) + r_{YY}(\tau) + j\left\{r_{YX}(\tau) - r_{XY}(\tau)\right\}$$

with $r_{ZZ}(\tau) = r_{ZZ}(-\tau)^*$.

c) Covariance Function

$$c_{ZZ}(\tau) = c_{XX}(\tau) + c_{YY}(\tau) + j\{c_{YX}(\tau) - c_{XY}(\tau)\}$$

with $c_{ZZ}(\tau) = c_{ZZ}(-\tau)^*$.

d) Variance Function

$$\sigma_Z^2 = \mathrm{Var}(Z_t) = c_{ZZ}(0) = c_{XX}(0) + c_{YY}(0) = \sigma_X^2 + \sigma_Y^2 = const.$$

as $c_{YX}(0) = c_{XY}(0)$.

e) Correlation Function

$$\rho_{ZZ}(\tau) = \frac{c_{ZZ}(\tau)}{\sigma_Z^2} = \frac{c_{ZZ}(\tau)}{c_{ZZ}(0)} \quad \text{with} \quad \rho_{ZZ}(\tau) = \rho_{ZZ}(-\tau)^*.$$

f) Cross Second Order Moment Function

$$r_{ZW}(\tau) = \mathrm{E}\left(Z_{t+\tau} W_t^*\right) = \mathrm{E}\left((X_{t+\tau} + jY_{t+\tau})(U_t - jV_t)\right)$$

$$= r_{XU}(\tau) + r_{YV}(\tau) + j\left\{r_{YU}(\tau) - r_{XV}(\tau)\right\}$$

with $r_{ZW}(\tau) = r_{WZ}(-\tau)^*$.

g) Cross Covariance Function

$$c_{ZW}(\tau) = c_{XU}(\tau) + c_{YV}(\tau) + j\left\{c_{YU}(\tau) - c_{XV}(\tau)\right\}$$

with $c_{ZW}(\tau) = c_{WZ}(-\tau)^*$.

Properties of the Moment Functions

(1) $\left|r_{zz}(\tau)\right| \leq r_{zz}(0), \ \left|c_{zz}(\tau)\right| \leq c_{zz}(0), \ \left|\rho_{zz}(\tau)\right| \leq 1.$

(2) $r_{zz}(\tau), c_{zz}(\tau), \rho_{zz}(\tau)$
are so-called positive semi-definite functions.

(3) $\left|r_{zw}(\tau)\right| \leq \sqrt{r_{zz}(0) \cdot r_{ww}(0)} \leq \frac{1}{2}\left(r_{zz}(0) + r_{ww}(0)\right),$

$$\left|c_{zw}(\tau)\right| \leq \sqrt{c_{zz}(0) \cdot c_{ww}(0)} \leq \frac{1}{2}\left(c_{xx}(0) + c_{ww}(0)\right).$$

Remarks:

Two stationary complex valued stochastic processes $(Z_t)_{t\in\mathcal{T}}$ and $(W_t)_{t\in\mathcal{T}}$ are said to be

- uncorrelated, if for all τ holds

  $$c_{ZW}(\tau)=\mathrm{E}\left((Z_{t+\tau}-\mu_Z)(W_t-\mu_W)^*\right)=\mathrm{E}\left(Z_{t+\tau}W_t^*\right)-\mu_Z\mu_W^*=0$$

  or equivalently

  $$r_{ZW}(\tau)=\mathrm{E}\left(Z_{t+\tau}W_t^*\right)=\mathrm{E}\left(Z_{t+\tau}\right)\mathrm{E}\left(W_t^*\right)=\mu_Z\mu_W^*,$$

- orthogonal, if for all τ holds

  $$r_{ZW}(\tau)=\mathrm{E}\left(Z_{t+\tau}W_t^*\right)=0.$$

2.4 Stochastic Limiting Operations

2.4.1 Stochastic Continuity

A stochastic process $(X_t)_{t \in \mathcal{T}}$ is said to be mean square continuous if

$$\lim_{\tau \to 0} \mathrm{E}\left(\left| X_{t+\tau} - X_t \right|^2 \right) = 0$$

holds for all $t \in \mathcal{T}$ and we write $\underset{\tau \to 0}{\mathrm{l.i.m.}}\, X_{t+\tau} = X_t$.

Theorem:

A stochastic process $(X_t)_{t \in \mathcal{T}}$ is mean square continuous if its second order moment function $r_{XX}(t_1,t_2)$ is continuous (in the ordinary sense) on the diagonal $t_1 = t_2 \in \mathcal{T}$.

Exercise 2.4-1:
Proof of the Theorem

Remarks:

- If $r_{XX}(t_1,t_2)$ is continuous on the diagonal $t_1 = t_2 \in \mathcal{T}$ one can show that $r_{XX}(t_1,t_2)$ is continuous everywhere.
- If a stochastic process $(X_t)_{t\in\mathcal{T}}$ is mean square continuous then its mean function $\mu_X(t)$ is continuous.
- A stationary stochastic process $(X_t)_{t\in\mathcal{T}}$ is mean square continuous if its second order moment function $r_{XX}(\tau)$ is continuous at $\tau = 0$.
- If $r_{XX}(\tau)$ is continuous at $\tau = 0$ one can show that $r_{XX}(\tau)$ is continuous everywhere.

2.4.2 Stochastic Differentiation

A stochastic process $(X_t)_{t\in\mathcal{T}}$ is said to be mean square differentiable if

$$\lim_{\tau\to 0} \mathrm{E}\left(\left|\frac{X_{t+\tau}-X_t}{\tau}-\dot{X}_t\right|^2\right)=0$$

holds for all $t\in\mathcal{T}$ and we write $\underset{\tau\to 0}{\mathrm{l.i.m.}}\,(X_{t+\tau}-X_t)/\tau=\dot{X}_t$.

Theorem:

A stochastic process $(X_t)_{t\in\mathcal{T}}$ is mean square differentiable if its second order moment function $r_{XX}(t_1,t_2)$ is twice continuously differentiable within $\mathcal{T}\times\mathcal{T}$ (sufficient condition).

Exercise 2.4-2:
Proof of the Theorem

Corollary:

(1) The mean function of $(\dot{X}_t)_{t\in\mathcal{T}}$ can be determined by

$$\mu_{\dot{X}}(t) = \mathrm{E}\left(\dot{X}_t\right) = \mathrm{E}\left(\underset{\tau\to 0}{\mathrm{l.i.m.}}\frac{X_{t+\tau} - X_t}{\tau}\right) = \lim_{\tau\to 0}\frac{\mathrm{E}\left(X_{t+\tau} - X_t\right)}{\tau}$$

$$= \lim_{\tau\to 0}\frac{\mathrm{E}\left(X_{t+\tau}\right) - \mathrm{E}\left(X_t\right)}{\tau} = \frac{d}{dt}\mathrm{E}\left(X_t\right) = \frac{d}{dt}\left(\mu_X(t)\right).$$

For a stationary stochastic process $(X_t)_{t\in\mathcal{T}}$ holds

$$\mu_{\dot{X}}(t) = \mathrm{E}\left(\dot{X}_t\right) = \frac{d}{dt}\mathrm{E}\left(X_t\right) = \frac{d}{dt}\left(\mu_X\right) \equiv 0.$$

(2) The second order moment function of $(\dot{X}_t)_{t\in\mathcal{T}}$ and the cross second order moment function of $(X_t)_{t\in\mathcal{T}}$ and $(\dot{X}_t)_{t\in\mathcal{T}}$ are given by

$$r_{\dot{X}\dot{X}}(t_1,t_2) = \mathrm{E}\left(\dot{X}_{t_1}\dot{X}_{t_2}\right) = \frac{\partial^2 r_{XX}(t_1,t_2)}{\partial t_1 \partial t_2},$$

$$r_{X\dot{X}}(t_1,t_2) = \mathrm{E}\left(X_{t_1}\dot{X}_{t_2}\right) = \frac{\partial r_{XX}(t_1,t_2)}{\partial t_2},$$

$$r_{\dot{X}X}(t_1,t_2) = \mathrm{E}\left(\dot{X}_{t_1}X_{t_2}\right) = \frac{\partial r_{XX}(t_1,t_2)}{\partial t_1}.$$

Exemplarily, the first expression can be derived by

$$\mathrm{E}\left(\underset{\tau\to 0}{\mathrm{l.i.m.}} \frac{\left(X_{t_1+\tau} - X_{t_1}\right)}{\tau} \cdot \underset{\delta\to 0}{\mathrm{l.i.m.}} \frac{\left(X_{t_2+\delta} - X_{t_2}\right)}{\delta} \right) =$$

$$= \lim_{\substack{\tau\to 0 \\ \delta\to 0}} \mathrm{E}\left(\frac{\left(X_{t_1+\tau} - X_{t_1}\right)}{\tau} \cdot \frac{\left(X_{t_2+\delta} - X_{t_2}\right)}{\delta} \right)$$

$$= \lim_{\substack{\tau\to 0 \\ \delta\to 0}} \frac{\mathrm{E}\left(X_{t_1+\tau} X_{t_2+\delta}\right) - \mathrm{E}\left(X_{t_1+\tau} X_{t_2}\right) - \mathrm{E}\left(X_{t_1} X_{t_2+\delta}\right) + \mathrm{E}\left(X_{t_1} X_{t_2}\right)}{\tau\,\delta}$$

$$= \lim_{\substack{\tau\to 0 \\ \delta\to 0}} \frac{r_{XX}(t_1+\tau, t_2+\delta) - r_{XX}(t_1, t_2+\delta) - r_{XX}(t_1+\tau, t_2) + r_{XX}(t_1, t_2)}{\tau\,\delta}$$

$$= \partial^2 r_{XX}(t_1, t_2) \Big/ \partial t_1 \partial t_2 .$$

For a stationary stochastic process $(X_t)_{t\in\mathcal{T}}$ we obtain

$$r_{\dot{X}X}(\tau) = r_{\dot{X}X}(t_1 - t_2) = \frac{\partial r_{XX}(t_1 - t_2)}{\partial t_1} = \frac{dr_{XX}(\tau)}{d\tau},$$

$$r_{X\dot{X}}(\tau) = r_{X\dot{X}}(t_1 - t_2) = \frac{\partial r_{XX}(t_1 - t_2)}{\partial t_2} = -\frac{dr_{XX}(\tau)}{d\tau},$$

$$r_{\dot{X}\dot{X}}(\tau) = r_{\dot{X}\dot{X}}(t_1 - t_2) = \frac{\partial^2 r_{XX}(t_1 - t_2)}{\partial t_1 \partial t_2} = -\frac{d^2 r_{XX}(\tau)}{d\tau^2}.$$

2.4.3 Stochastic Integration

A stochastic process $(X_t)_{t\in\mathcal{T}}$ is said to be mean square integrable if the limit in mean square

$$\underset{\max(t_i - t_{i-1})\to 0}{\text{l.i.m.}} \sum_{i=1}^{n} X_{t_i}(t_i - t_{i-1}) = \int_{T_1}^{T_2} X_t \, dt$$

exists for all $T_1, T_2 \in \mathcal{T}$.

Theorem:

A stochastic process $(X_t)_{t\in\mathcal{T}}$ is mean square integrable if

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} r_{XX}(t_1,t_2)\,dt_1 dt_2$$

exists as a Riemann double integral.

Exercise 2.4-3:
Proof of the Theorem

Remarks:

- To show the existence of the Riemann double integral

$$\int_{T_1}^{T_2}\int_{T_1}^{T_2} r_{XX}(t_1,t_2)\,dt_1\,dt_2$$

  it is sufficient to show that $r_{XX}(t_1,t_2)$ is continuous over the closed set $[T_1,T_2]\times[T_1,T_2]$.

- In applications one may also be concerned with integrals extending over infinite ranges. However, with obvious modifications, the above result can be extended to cover these more general cases.

2.5 Spectral Analysis of Stationary Processes

2.5.1 Spectral Density Function

A typical realization of a general continuous time zero mean stationary process will be neither periodic nor of bounded energy. Hence, it cannot be represented as a Fourier series or as a Fourier integral.

However, to overcome this difficulty, we define the new process X_t^T by

$$X_t^T = \begin{cases} X_t & -T/2 \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}.$$

Assuming X_t^T to be continuous, it can be expressed as

Fourier integral, i.e.

$$X_t^T = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^T(\omega)e^{j\omega t}d\omega \ \text{ with } \ X^T(\omega) = \int_{-\infty}^{\infty} X_t^T e^{-j\omega t}dt,$$

where $\left|X^T(\omega)\right|^2$ has the following physical interpretation.

$$\left|X^T(\omega)\right|^2 \frac{d\omega}{2\pi} = \begin{cases} \text{part of the total energy of } X_t^T \\ \text{contributed by components with} \\ \text{frequencies between } \omega, \omega + d\omega \end{cases}$$

Even though $\left|X^T(\omega)\right|^2$ may tend towards infinity as $T \to \infty$,

$$\left(\lim_{T\to\infty}\left|X^T(\omega)\right|^2 / T\right)\frac{d\omega}{2\pi} = \begin{cases} \text{part of the total power of } X_t^T \\ \text{contributed by components with} \\ \text{frequencies between } \omega, \omega + d\omega \end{cases}$$

may converge to a finite limit, where

$$\lim_{T\to\infty} \left| X^T(\omega) \right|^2 / T$$

would have an interpretation as a power density function.

Since the analysis above refers only to a single realization the value of the limit, if it exists, will change from one realization to another.

To construct a quantity which characterises the spectral properties of the whole stochastic process averaging of

$$\left| X^T(\omega) \right|^2 / T$$

over the different realizations seems to be natural.

Therefore, one defines

$$C_{XX}(\omega) = \lim_{T\to\infty}\left\{\mathrm{E}\left(\left|X^T(\omega)\right|^2 / T\right)\right\}.$$

When the limit exists, the function $C_{XX}(\omega)$ is called power spectral density function of the zero mean stationary process X_t.

The calculation of $C_{XX}(\omega)$ from the definition above would be an ambitious task. Fortunately the following basic result, known as Wiener-Khintchine Formula, provides an alternative way for calculating $C_{XX}(\omega)$.

Theorem:

Let $(X_t)_{t\in\mathcal{T}}$ be a zero mean continuous time stationary stochastic process with power spectral density function $C_{XX}(\omega)$ and covariance function $c_{XX}(\tau)$. Then $C_{XX}(\omega)$ is the Fourier transform of $c_{XX}(\tau)$, i.e.

$$C_{XX}(\omega) = \int_{-\infty}^{\infty} c_{XX}(\tau) e^{-j\omega\tau} d\tau.$$

Remark:

The power spectral density function $C_{XX}(\omega)$ exists for all ω if the covariance function $c_{XX}(\tau)$ possesses a Fourier transform. Hence, a sufficient condition is given by

$$\int_{-\infty}^{\infty} \left| c_{XX}(\tau) \right| d\tau < \infty.$$

Exercise 2.5-1:
Proof of the Theorem

Properties of the Power Spectral Density Function

(1) $C_{XX}(\omega) \geq 0$ for all ω since $\left|X^T(\omega)\right|^2 \geq 0.$

(2) $C_{XX}(\omega)$ is real since $c_{XX}(\tau) = c_{XX}(-\tau)^*.$

(3) For real valued processes

$C_{XX}(\omega) = C_{XX}(-\omega)$ since $c_{XX}(\tau) = c_{XX}(-\tau).$

(4) $Y_t = a\,X_t,\;\; c_{YY}(\tau) = a^2 c_{XX}(\tau),\;\; C_{YY}(\omega) = a^2 C_{XX}(\omega).$

(5) $Y_t = \dot{X}_t,\;\; c_{YY}(\tau) = -\dfrac{d^2 c_{XX}(\tau)}{d\tau^2},\;\; C_{YY}(\omega) = \omega^2 C_{XX}(\omega).$

Remarks:

- For a continuous time stationary stochastic process with periodic components the Fourier transform

  $$C_{XX}(\omega) = \int_{-\infty}^{\infty} c_{XX}(\tau) e^{-j\omega\tau} d\tau$$

  has to exist in consideration of generalized functions.

- The power spectral density function of a continuous time stationary stochastic process with expected value μ_X is given by

  $$R_{XX}(\omega) = \int_{-\infty}^{\infty} r_{XX}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \left(c_{XX}(\tau) + \mu_X^2 \right) e^{-j\omega\tau} d\tau$$

  $$= C_{XX}(\omega) + 2\pi \mu_X^2 \delta(\omega).$$

If $c_{XX}(\tau)$ is absolutely integrable and if $c_{XX}(\tau)$ is continuous at $\tau = 0$ then $c_{XX}(\tau)$ can be expressed as the inverse Fourier transform of $C_{XX}(\omega)$, i.e.

$$c_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{XX}(\omega) e^{j\omega\tau} \, d\omega.$$

Now, setting $\tau = 0$, we obtain

$$\sigma_X^2 = c_{XX}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C_{XX}(\omega) \, d\omega$$

which represents the total power of the process, i.e. the power contributed by all frequency components.

Hence, the variance of a zero mean stationary stochastic process is a measure of its total power.

Let $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ be zero mean continuous time stationary stochastic processes. The Fourier transform of the cross covariance function

$$C_{XY}(\omega)=\int_{-\infty}^{\infty} c_{XY}(\tau)e^{-j\omega\tau}d\tau$$

is called cross power spectral density function.

If $c_{XY}(\tau)$ is absolutely integrable and if $c_{XY}(\tau)$ is continuous then $c_{XY}(\tau)$ can be expressed as the inverse Fourier transform of $C_{XY}(\omega)$, i.e.

$$c_{XY}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C_{XY}(\omega)e^{j\omega\tau}d\omega.$$

Properties of the Cross Power Spectral Density Function

For real valued processes $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ holds

(1) $C_{XY}(-\omega) = C_{XY}(\omega)^*$,

(2) $C_{XY}(\omega) = C_{YX}(\omega)^*$.

Remarks:

- For continuous time stationary stochastic processes with periodic components the Fourier transform

$$C_{XY}(\omega) = \int_{-\infty}^{\infty} c_{XY}(\tau) e^{-j\omega\tau} d\tau$$

has to exist in consideration of generalized functions.

- The cross power spectral density function of the continuous time stationary stochastic processes $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ with the expected values μ_X and μ_Y respectively is given by

$$R_{XY}(\omega)=\int_{-\infty}^{\infty} r_{XY}(\tau)e^{-j\omega\tau}\,d\tau$$

$$=\int_{-\infty}^{\infty}\left(c_{XY}(\tau)+\mu_X\mu_Y\right)e^{-j\omega\tau}\,d\tau$$

$$=C_{XY}(\omega)+2\pi\,\mu_X\mu_Y\,\delta(\omega).$$

Analogue to the continuous time case, the relationship among the covariance function $c_{XX}(\tau)$ and the power spectral density function $C_{XX}(\Omega)$ of a discrete time zero mean stationary stochastic process $(X_t)_{t\in\mathcal{T}}$ can be defined by

$$C_{XX}(\Omega)=\sum_{\tau=-\infty}^{\infty} c_{XX}(\tau)e^{-j\Omega\tau}$$

and

$$c_{XX}(\tau)=\frac{1}{2\pi}\int_{-\pi}^{\pi} C_{XX}(\Omega)e^{j\Omega\tau}\,d\Omega.$$

Remarks:

- For a discrete time stationary stochastic process with periodic components the Fourier transform

$$C_{XX}(\Omega)=\sum_{\tau=-\infty}^{\infty} c_{XX}(\tau)e^{-j\Omega\tau}$$

has to exist in consideration of generalized functions.

- The power spectral density function of a discrete time stationary stochastic process with expected value μ_X is given by

$$R_{XX}(\Omega)=\sum_{\tau=-\infty}^{\infty} r_{XX}(\tau)e^{-j\Omega\tau}=\sum_{\tau=-\infty}^{\infty}\left(c_{XX}(\tau)+\mu_X^2\right)e^{-j\Omega\tau}$$

$$=C_{XX}(\Omega)+2\pi\mu_X^2\eta(\Omega)$$

with $\eta(\Omega) = \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$.

Correspondingly, the cross covariance function $c_{XY}(\tau)$ and the cross power spectral density function $C_{XY}(\Omega)$ of the zero mean discrete time stationary stochastic processes $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$ are linked up as follows.

$$C_{XY}(\Omega) = \sum_{\tau=-\infty}^{\infty} c_{XY}(\tau) e^{-j\Omega\tau}$$

and

$$c_{XY}(\tau) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{XY}(\Omega) e^{j\Omega\tau} d\Omega.$$

Moreover, let X denote a discrete time process that is obtained by sampling the continuous time zero mean stationary stochastic process X^c equidistantly in time with sampling period Δ, i.e.

$$X_t = X^c_{\Delta t} \quad \text{for} \quad t = \ldots -1,0,1,\ldots \quad \text{and} \quad \Delta t \in \mathbb{R}.$$

The covariance function and the power spectral density function of the discrete and continuous time stationary stochastic processes are then related by

$$c_{XX}(\tau) = \mathrm{Cov}(X_{t+\tau}, X_t) = \mathrm{Cov}(X^c_{\Delta(t+\tau)}, X^c_{\Delta t}) = c_{X^c X^c}(\Delta\tau)$$

$$\text{for} \quad t,\tau = \ldots -1,0,1,\ldots$$

and

$$C_{XX}(\Omega) = \sum_{\tau=-\infty}^{\infty} c_{XX}(\tau) e^{-j\Omega\tau} = \sum_{\tau=-\infty}^{\infty} c_{X^c X^c}(\Delta\tau) e^{-j\Omega\tau}$$

$$= \sum_{\tau=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} C_{X^c X^c}(\omega) e^{j\omega\Delta\tau} d\omega \right) e^{-j\Omega\tau}$$

$$= \int_{-\infty}^{\infty} C_{X^c X^c}(\omega) \frac{1}{2\pi} \sum_{\tau=-\infty}^{\infty} e^{j(\omega-\Omega/\Delta)\Delta\tau} d\omega$$

$$= \int_{-\infty}^{\infty} C_{X^c X^c}(\omega) \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} \delta\big(\omega - (\Omega - 2\pi k)/\Delta\big) d\omega$$

$$= \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} C_{X^c X^c}(\omega) \delta\big(\omega - (\Omega - 2\pi k)/\Delta\big) d\omega$$

$$= \frac{1}{\Delta} \sum_{k=-\infty}^{\infty} C_{X^c X^c}\big((\Omega - 2\pi k)/\Delta\big).$$

2.5.2 Spectral Representation of Stationary Processes

Continuous time stationary processes

$$X_t = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} dZ_X(\omega) \quad \text{with} \quad \mathrm{E}\left(dZ_X(\omega)\right) = 2\pi\,\mu_X\,\delta(\omega)\,d\omega$$

and

$$\mathrm{Cov}\left(dZ_X(\omega), dZ_X(\lambda)\right) = 2\pi\, C_{XX}(\omega)\,\delta(\omega-\lambda)\,d\omega\, d\lambda,$$

so that the mean value of X_t can be also derived by

$$\mathrm{E}(X_t) = \mathrm{E}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} dZ_X(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\,\mathrm{E}\left(dZ_X(\omega)\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t}\, 2\pi\,\mu_X\,\delta(\omega)\,d\omega = \mu_X$$

Furthermore, the relationship between the covariance and spectral density function can be approved

$$
\begin{aligned}
c_{XX}(\tau) &= \mathrm{Cov}(X_{t+\tau}, X_t) \\
&= \mathrm{Cov}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega(t+\tau)}\,dZ_X(\omega), \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\lambda t}\,dZ_X(\lambda)\right) \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega\tau} e^{j(\omega-\lambda)t}\,\mathrm{Cov}\big(dZ_X(\omega), dZ_X(\lambda)\big) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega\tau} e^{j(\omega-\lambda)t}\,C_{XX}(\omega)\,\delta(\omega-\lambda)\,d\omega\,d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} C_{XX}(\omega)\,e^{j\omega\tau}\,d\omega.
\end{aligned}
$$

Discrete time stationary processes

$$X_t = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\, dZ_X(\Omega) \quad \text{with} \quad \mathrm{E}\left(dZ_X(\Omega)\right) = 2\pi\,\mu_X\,\eta(\Omega)\,d\Omega$$

and

$$\mathrm{Cov}\left(dZ_X(\Omega), dZ_X(\Lambda)\right) = 2\pi\, C_{XX}(\Omega)\,\eta(\Omega-\Lambda)\,d\Omega\, d\Lambda$$

so that the mean value of X_t can be also derived by

$$\mathrm{E}(X_t) = \mathrm{E}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\, dZ_X(\Omega)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\,\mathrm{E}\left(dZ_X(\Omega)\right)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\Omega t}\, 2\pi\,\mu_X\,\eta(\Omega)\,d\Omega = \mu_X .$$

Furthermore, the relationship between the covariance and spectral density function can be approved.

$$
\begin{aligned}
c_{XX}(\tau) &= \mathrm{Cov}(X_{t+\tau}, X_t) \\
&= \mathrm{Cov}\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(t+\tau)} dZ_X(\Omega), \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\lambda t} dZ_X(\Lambda) \right) \\
&= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{j\Omega\tau} e^{j(\Omega-\Lambda)t} \mathrm{Cov}\left(dZ_X(\Omega), dZ_X(\Lambda)\right) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} e^{j\Omega\tau} e^{j(\Omega-\Lambda)t} C_{XX}(\Omega) \eta(\Omega - \Lambda) d\Omega\, d\Lambda \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} C_{XX}(\Omega) e^{j\Omega\tau} d\Omega
\end{aligned}
$$

2.6 Systems with Stochastic Inputs

Let T denote an operator that assigns to each function x_t a function y_t, i.e. $y_t = \mathrm{T}[x_t]$.

The operator T is called

- deterministic, if $x_{t,1} = x_{t,2} \;\; \forall t \;\Rightarrow\; \mathrm{T}[x_{t,1}] = \mathrm{T}[x_{t,2}] \;\; \forall t.$
- memoryless, if a function g exists such that $y_t = \mathrm{T}[x_t] = g(x_t)$.
- time invariant, if $y_{t+\tau} = \mathrm{T}[x_{t+\tau}]$.
- linear, if it is homogenous, additive and continuous, i.e.

$$y_t = \mathrm{T}[x_t] = \mathrm{T}\left[\sum_{i=1}^{\infty} a_i x_{t,i}\right] = \sum_{i=1}^{\infty} a_i \mathrm{T}[x_{t,i}] = \sum_{i=1}^{\infty} a_i y_{t,i}.$$

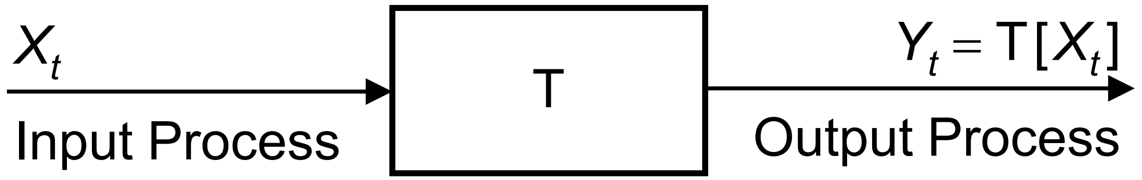
2.6.1 Transformation of Stochastic Processes

Let $(X_t)_{t\in\mathcal{T}}$ denote a stochastic process. An operator T assigns to each realization x_t of $(X_t)_{t\in\mathcal{T}}$ a time dependent function y_t which is a realization of a new stochastic process $(Y_t)_{t\in\mathcal{T}}$. Therefore,

$$Y_t = \mathrm{T}[X_t]$$

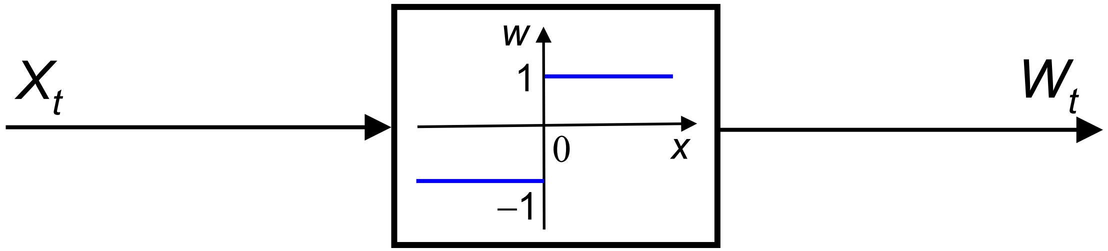
defines a so called transformed stochastic process.

Systems Theoretic Interpretation:



2.6.2 Memoryless Systems

Hard limiter $g(x) = \text{sgn}(x): \quad W_t = \text{sgn}(X_t)$



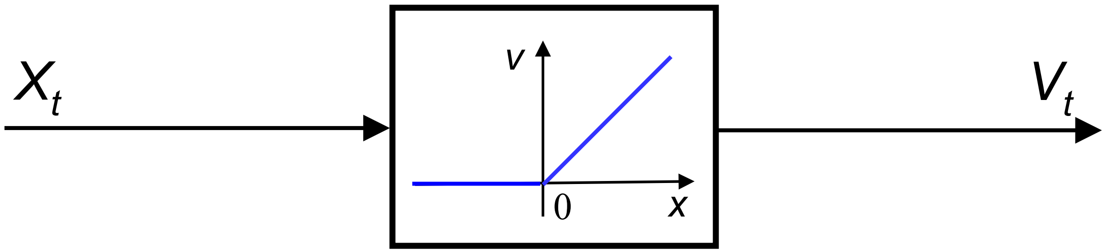
$$f_W(w;t) = \left(1 - F_X(0;t)\right)\delta(w-1) + F_X(0;t)\delta(w+1),$$

$$\mu_W(t) = 1 - 2F_X(0;t),$$

$$r_{WW}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \text{sgn}(x_1)\,\text{sgn}(x_2)\, f_{X_1X_2}(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

$$= P\left(X_{t_1}X_{t_2} > 0\right) - P\left(X_{t_1}X_{t_2} \le 0\right).$$

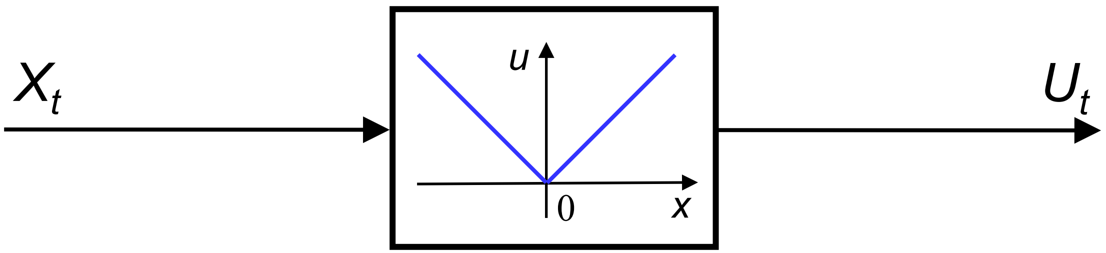
Half wave rectifier $g(x) = \frac{1}{2}(x + |x|): \quad V_t = \frac{1}{2}(X_t + |X_t|)$



$$f_V(v;t) = \begin{cases} f_X(v;t) & v > 0 \\ F_X(0;t)\,\delta(v) & v = 0, \\ 0 & v < 0 \end{cases} \qquad \mu_V(t) = \int_0^\infty x f_X(x;t)\,dx,$$

$$r_{VV}(t_1,t_2) = \int_0^\infty \int_0^\infty x_1 x_2 f_{X_1X_2}(x_1,x_2;t_1,t_2)\,dx_1 dx_2.$$

Full wave rectifier $g(x)=|x|: \quad U_t = |X_t|$

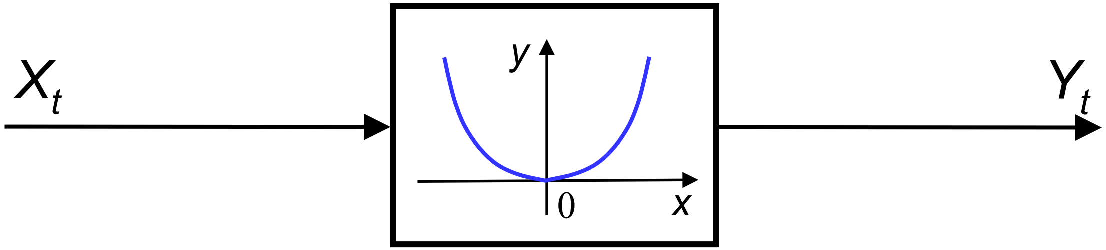


$$f_U(u;t)=\begin{cases} f_X(u;t)+f_X(-u;t) & u \geq 0 \\ 0 & u<0 \end{cases},$$

$$\mu_U(t)=\int_{-\infty}^{\infty}|x| f_X(x;t)\,dx=\int_0^{\infty}u\big(f_X(u;t)+f_X(-u;t)\big)\,du,$$

$$r_{UU}(t_1,t_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|x_1x_2| f_{X_1X_2}(x_1,x_2;t_1,t_2)\,dx_1dx_2.$$

Square-law detector $g(x) = x^2$: $Y_t = X_t^2$



$$f_Y(y;t) = \begin{cases} \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y};t) + f_X(-\sqrt{y};t)\right) & y \geq 0 \\ 0 & y < 0 \end{cases},$$

$$\mu_Y(t) = \int_{-\infty}^{\infty} x^2 f_X(x,t)\,dx = \frac{1}{2}\int_0^{\infty} \sqrt{y}\left(f_X(\sqrt{y};t) + f_X(-\sqrt{y};t)\right)dy,$$

$$r_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 x_2^2 f_{X_1X_2}(x_1,x_2;t_1,t_2)\,dx_1 dx_2.$$

2.6.3 Linear Systems

Theorem:

1) The order of application of the expectation operator E and the stable LTI system operator T can be interchanged.
2) Let $(X_t)_{t\in\mathcal{T}}$ be a stationary stochastic process and T an operator of a stable LTI system then the stochastic process $Y_t = \mathrm{T}[X_t]$ is stationary.
3) Let $(X_t)_{t\in\mathcal{T}}$ be a wide sense stationary stochastic process and T an operator of a stable LTI system then the stochastic process $Y_t = \mathrm{T}[X_t]$ is wide sense stationary.

LTI = Linear Time Invariant

Continuous time LTI Systems

Let $(X_t)_{t\in\mathcal{T}}$ be a continuous time stationary stochastic input process of a stable and continuous time LTI system with impulse response h_t. Thus, the continuous time stationary output process $(Y_t)_{t\in\mathcal{T}}$ is given by

$$Y_t = \mathrm{T}[X_t] = \int_{-\infty}^{\infty} h_{t'} X_{t-t'} dt' = \int_{-\infty}^{\infty} h_{t-t'} X_{t'} dt',$$

where the impulse response satisfies

$$\int_{-\infty}^{\infty} |h_t| dt < \infty$$

and therefore the transfer function

$$H(\omega) = \int_{-\infty}^{\infty} h_t e^{-j\omega t} dt$$

exists.

Using the spectral representation of a continuous time stationary stochastic processes we obtain

$$Y_t = \int_{-\infty}^{\infty} h_{t'} X_{t-t'} dt' = \int_{-\infty}^{\infty} h_{t'} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega(t-t')} dZ_X(\omega) \right) dt'$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} \left(\int_{-\infty}^{\infty} h_{t'} e^{-j\omega t'} dt' \right) dZ_X(\omega)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} H(\omega) dZ_X(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} dZ_Y(\omega),$$

where

$$dZ_Y(\omega) = H(\omega) dZ_X(\omega)$$

with

$$\mathrm{E}\left(dZ_Y(\omega)\right) = H(\omega)\mathrm{E}\left(dZ_X(\omega)\right) = 2\pi\, \mu_X\, H(\omega)\delta(\omega)d\omega.$$

Mean of the output process

$$\mathrm{E}(Y_t) = \mathrm{E}\left(\int_{-\infty}^{\infty} h_{t'} X_{t-t'}\, dt'\right)$$

$$= \int_{-\infty}^{\infty} h_{t'} \mathrm{E}(X_{t-t'})\, dt' = \mu_X \int_{-\infty}^{\infty} h_{t'}\, dt' = \mu_X H(0)$$

Covariance function of the output process

$$c_{YY}(\tau) = \mathrm{Cov}(Y_{t+\tau}, Y_t)$$

$$= \mathrm{Cov}\left(\int_{-\infty}^{\infty} h_{t'} X_{t+\tau-t'}\, dt', \int_{-\infty}^{\infty} h_{t''} X_{t-t''}\, dt''\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{t'} h_{t''} \mathrm{Cov}(X_{t+\tau-t'}, X_{t-t''})\, dt' dt''$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_{t'} h_{t''} c_{XX}(\tau - t' + t'')\, dt' dt''$$

Power spectral density of the output process

$$C_{YY}(\omega) = \int_{-\infty}^{\infty} c_{YY}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{t'} h_{t''} c_{XX}(\tau - t' + t'') dt' dt'' \right) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{t'} h_{t''} \left(\int_{-\infty}^{\infty} c_{XX}(\tau - t' + t'') e^{-j\omega\tau} d\tau \right) dt' dt''$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_{t'} h_{t''} C_{XX}(\omega) e^{-j\omega(t' - t'')} dt' dt''$$

$$= \int_{-\infty}^{\infty} h_{t'} e^{-j\omega t'} dt' \cdot \int_{-\infty}^{\infty} h_{t''} e^{j\omega t''} dt'' \cdot C_{XX}(\omega)$$

$$= H(\omega) H(\omega)^* C_{XX}(\omega) = \left| H(\omega) \right|^2 C_{XX}(\omega)$$

or

$$\begin{aligned}\mathrm{Cov}\left(dZ_Y(\omega),dZ_Y(\lambda)\right) &= H(\omega)H(\lambda)^*\,\mathrm{Cov}\left(dZ_X(\omega),dZ_X(\lambda)\right)\\ &= 2\pi\left|H(\omega)\right|^2 C_{XX}(\omega)\delta(\omega-\lambda)\,d\omega\, d\lambda\\ &= 2\pi C_{YY}(\omega)\delta(\omega-\lambda)\,d\omega\, d\lambda\end{aligned}$$

$$\Rightarrow \quad C_{YY}(\omega) = \left|H(\omega)\right|^2 C_{XX}(\omega)$$

Cross covariance function of input & output process

$$\begin{aligned}c_{XY}(\tau) &= \mathrm{Cov}(X_{t+\tau},Y_t) = \mathrm{Cov}\left(X_{t+\tau},\int_{-\infty}^{\infty} h_{t'}X_{t-t'}\,dt'\right)\\ &= \int_{-\infty}^{\infty} h_{t'}\,\mathrm{Cov}(X_{t+\tau},X_{t-t'})\,dt' = \int_{-\infty}^{\infty} h_{t'}\,c_{XX}(\tau+t')\,dt'\end{aligned}$$

Cross spectral density function of input & output process

$$C_{XY}(\omega) = \int_{-\infty}^{\infty} c_{XY}(\tau) e^{-j\omega\tau} d\tau = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h_{t'} c_{XX}(\tau + t') dt' \right) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} h_{t'} \left(\int_{-\infty}^{\infty} c_{XX}(\tau + t') e^{-j\omega\tau} d\tau \right) dt'$$

$$= \int_{-\infty}^{\infty} h_{t'} e^{j\omega t'} dt' \cdot C_{XX}(\omega) = H(\omega)^* C_{XX}(\omega)$$

or

$$\mathrm{Cov}\left(dZ_X(\omega), dZ_Y(\lambda)\right) = H(\lambda)^* \,\mathrm{Cov}\left(dZ_X(\omega), dZ_X(\lambda)\right)$$

$$= 2\pi H(\omega)^* C_{XX}(\omega) \delta(\omega - \lambda) d\omega d\lambda$$

$$= 2\pi C_{XY}(\omega) \delta(\omega - \lambda) d\omega d\lambda$$

$$\Rightarrow \quad C_{XY}(\omega) = H(\omega)^* C_{XX}(\omega)$$

Discrete time LTI Systems

Consider a discrete time stationary stochastic process $(X_t)_{t\in\mathcal{T}}$ to be the input of a stable and discrete time LTI system with impulse response h_t. Hence, the discrete time stationary output process $(Y_t)_{t\in\mathcal{T}}$ is given by

$$Y_t = \mathrm{T}[X_t] = \sum_{t'=-\infty}^{\infty} h_{t'} X_{t-t'} = \sum_{t'=-\infty}^{\infty} h_{t-t'} X_t,$$

where the impulse response satisfies

$$\sum_{t=-\infty}^{\infty} |h_t| < \infty$$

and therefore the transfer function

$$H(\Omega) = \sum_{t=-\infty}^{\infty} h_t e^{-j\Omega t}$$

exists.

Exploiting the spectral representation of a discrete time stationary stochastic processes we obtain

$$Y_t = \sum_{t'=-\infty}^{\infty} h_{t'} X_{t-t'} = \sum_{t'=-\infty}^{\infty} h_{t'} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega(t-t')} dZ_X(\Omega) \right)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega t} \left(\sum_{t'=-\infty}^{\infty} h_{t'} e^{-j\Omega t'} \right) dZ_X(\Omega)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega t} H(\Omega) dZ_X(\Omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\Omega t} dZ_Y(\Omega),$$

where

$$dZ_Y(\Omega) = H(\Omega) dZ_X(\Omega)$$

with

$$\mathrm{E}\left(dZ_Y(\Omega)\right) = H(\Omega)\mathrm{E}\left(dZ_X(\Omega)\right) = 2\pi \mu_X H(\Omega) \eta(\Omega) d\Omega.$$

Mean of the output process

$$\mathrm{E}(Y_t) = \mathrm{E}\left(\sum_{t'=-\infty}^{\infty} h_{t'} X_{t-t'}\right) = \sum_{t'=-\infty}^{\infty} h_{t'} \mathrm{E}(X_{t-t'}) = \mu_X \sum_{t'=-\infty}^{\infty} h_{t'}$$

$$= \mu_X H(0)$$

Covariance function of the output process

$$c_{YY}(\tau) = \mathrm{Cov}(Y_{t+\tau}, Y_t) = \mathrm{Cov}\left(\sum_{t'=-\infty}^{\infty} h_{t'} X_{t+\tau-t'}, \sum_{t''=-\infty}^{\infty} h_{t''} X_{t-t''}\right)$$

$$= \sum_{t'=-\infty}^{\infty} \sum_{t''=-\infty}^{\infty} h_{t'} h_{t''} \mathrm{Cov}(X_{t+\tau-t'}, X_{t-t''})$$

$$= \sum_{t'=-\infty}^{\infty} \sum_{t''=-\infty}^{\infty} h_{t'} h_{t''} c_{XX}(\tau - t' + t'')$$

Power spectral density of the output process

$$C_{YY}(\Omega) = \sum_{\tau=-\infty}^{\infty} c_{YY}(\tau) e^{-j\Omega\tau}$$

$$= \sum_{\tau=-\infty}^{\infty} \left(\sum_{t'=-\infty}^{\infty} \sum_{t''=-\infty}^{\infty} h_{t'} h_{t''} c_{XX}(\tau - t' + t'') \right) e^{-j\Omega\tau}$$

$$= \sum_{t'=-\infty}^{\infty} \sum_{t''=-\infty}^{\infty} h_{t'} h_{t''} \left(\sum_{\tau=-\infty}^{\infty} c_{XX}(\tau - t' + t'') e^{-j\Omega\tau} \right)$$

$$= \sum_{t'=-\infty}^{\infty} \sum_{t''=-\infty}^{\infty} h_{t'} h_{t''} C_{XX}(\Omega) e^{-j\Omega(t'-t'')}$$

$$= \sum_{t'=-\infty}^{\infty} h_{t'} e^{-j\Omega t'} \cdot \sum_{t''=-\infty}^{\infty} h_{t''} e^{j\Omega t''} \cdot C_{XX}(\Omega) = \left| H(\Omega) \right|^2 C_{XX}(\Omega)$$

or

$$\mathrm{Cov}\left(dZ_Y(\Omega), dZ_Y(\Lambda)\right) = H(\Omega)H(\Lambda)^* \mathrm{Cov}\left(dZ_X(\Omega), dZ_X(\Lambda)\right)$$

$$= 2\pi \left|H(\Omega)\right|^2 C_{XX}(\Omega)\eta(\Omega - \Lambda)\, d\Omega\, d\Lambda$$

$$= 2\pi C_{YY}(\Omega)\eta(\Omega - \Lambda)\, d\Omega\, d\Lambda$$

$$\Rightarrow \quad C_{YY}(\Omega) = \left|H(\Omega)\right|^2 C_{XX}(\Omega)$$

Cross covariance function of input & output process

$$c_{XY}(\tau) = \mathrm{Cov}(X_{t+\tau}, Y_t) = \mathrm{Cov}\left(X_{t+\tau}, \sum_{t'=-\infty}^{\infty} h_{t'} X_{t-t'} \right)$$

$$= \sum_{t'=-\infty}^{\infty} h_{t'} \mathrm{Cov}(X_{t+\tau}, X_{t-t'}) = \sum_{t'=-\infty}^{\infty} h_{t'} c_{XX}(\tau + t')$$

Cross spectral density function of input & output process

$$C_{XY}(\Omega) = \sum_{\tau=-\infty}^{\infty} c_{XY}(\tau) e^{-j\Omega\tau} = \sum_{\tau=-\infty}^{\infty} \left(\sum_{t'=-\infty}^{\infty} h_{t'} c_{XX}(\tau + t') \right) e^{-j\Omega\tau}$$

$$= \sum_{t'=-\infty}^{\infty} h_{t'} \left(\sum_{\tau=-\infty}^{\infty} c_{XX}(\tau + t') e^{-j\Omega\tau} \right) = \sum_{t'=-\infty}^{\infty} h_{t'} e^{j\Omega t'} \cdot C_{XX}(\Omega)$$

$$= H(\Omega)^* C_{XX}(\Omega)$$

or

$$\mathrm{Cov}\left(dZ_X(\Omega), dZ_Y(\Lambda)\right) = H(\lambda)^* \mathrm{Cov}\left(dZ_X(\Omega), dZ_X(\Lambda)\right)$$

$$= 2\pi H(\Omega)^* C_{XX}(\Omega) \eta(\Omega - \Lambda) d\Omega \, d\Lambda$$

$$= 2\pi C_{XY}(\Omega) \eta(\Omega - \Lambda) d\Omega \, d\Lambda$$

$$\Rightarrow \quad C_{XY}(\Omega) = H(\Omega)^* C_{XX}(\Omega)$$

2.7 Special Discrete-time Parameter Models

2.7.1 Purely Random Processes, White Noise

The process (X_t), $t = \ldots, -2, -1, 0, 1, 2, \ldots$ is called purely random process if it consists of a sequence of uncorrelated random variables.

For such a process to be wide-sense stationary we require only that

$$\mathrm{E}(X_t) = \mu \quad \text{and} \quad \mathrm{Var}(X_t) = \mathrm{E}(X_t - \mu)^2 = \sigma^2 \quad \forall t.$$

The covariance function given by

$$c_{XX}(\tau) = \mathrm{Cov}(X_{t+\tau} X_t) = \mathrm{E}\big((X_{t+\tau} - \mu)(X_t - \mu)\big) = \sigma^2 \delta_\tau$$

is then automatically a function of τ only.

In the sequel, we denote a stationary purely random process with $\mu = 0$ by (Z_t) and call it white noise.

The term white noise is to be understood in a figurative sense for white light, in which different optical frequency components are superimposed to form a white color impression. However, light that is subjectively perceived as white by humans does not have a constant power density spectrum.

Although the white noise model appears to be highly artificial (memoryless processes hardly occur in practice) it is nevertheless important since it provides us a basic building block for the construction of more complicated models.

2.7.2 Auto-Regressive (AR)-Processes

We say that (X_t) is an auto-regressive process of order p (denoted by AR(p)) if it satisfies the difference equation

$$X_t + a_1 X_{t-1} + \ldots + a_p X_{t-p} = X_t + \sum_{n=1}^{p} a_n X_{t-n} = Z_t,$$

where $a_1, a_2, \ldots, a_p$ are constants and (Z_t) is white noise.

Using the backward shift operator B the difference equation can be written more concisely in the form

$$\left(1 + \sum_{n=1}^{p} a_n B^n\right) X_t = a(B) X_t = Z_t .$$

The formal solution of the equation can be expressed by

$$X_t = \underbrace{A_1\lambda_1^t + A_2\lambda_2^t + \ldots + A_p\lambda_p^t}_{\text{solution of the homogeneous equation } a(B)X_t=0} + \underbrace{a^{-1}(B)Z_t}_{\text{particular solution of } a(B)X_t=Z_t} ,$$

where $\lambda_1, \lambda_2, ..., \lambda_p$ are the roots (assuming distinct roots) of the polynomial $\alpha(z) = a(z^{-1})z^p = z^p + \sum_{n=1}^{p} a_n z^{p-n}$.

Hence, asymptotic stationarity is obtained if $|\lambda_n| < 1$ for $n = 1, ..., p$, i.e. the roots of $\alpha(z)$ must lie inside the unit circle.

Let (X_t) be stationary, then we may ignore the solution of $a(B)X_t = 0$ (which will decay to zero), and the steady state solution can be written as

$$X_t = a^{-1}(B)Z_t = \left(\sum_{n=0}^{\infty} h_n B^n\right) Z_t = \sum_{n=0}^{\infty} h_n Z_{t-n}.$$

Now, multiplying both sides of the difference equation

$$X_t + \sum_{n=1}^{p} a_n X_{t-n} = Z_t \quad \left(\mathrm{E}(Z_t) = 0 \Rightarrow \mathrm{E}(X_t) = 0\right)$$

by X_{t-m} from the right and taking expectations we obtain

$$\mathrm{E}\left(\left(X_t + \sum_{n=1}^{p} a_n X_{t-n}\right) X_{t-m}\right) = \mathrm{E}(Z_t X_{t-m})$$

$$\mathrm{E}(X_t X_{t-m}) + \sum_{n=1}^{p} a_n \mathrm{E}(X_{t-n} X_{t-m}) = \mathrm{E}(Z_t X_{t-m})$$

$$c_{XX}(m) + \sum_{n=1}^{p} a_n c_{XX}(m-n) = c_{ZX}(m) = \sigma_Z^2 \delta_m = \begin{cases} \sigma_Z^2 & m = 0 \\ 0 & m > 0 \end{cases}.$$

This set of equations is known as Yule-Walker equations.

For $m = 1, 2, \ldots, p$ the Yule-Walker equations can be expressed by

$$\begin{pmatrix} c_{XX}(0) & c_{XX}(1) & \cdots & c_{XX}(p-1) \\ c_{XX}(1) & c_{XX}(0) & \cdots & c_{XX}(p-2) \\ \vdots & \vdots & \ddots & \vdots \\ c_{XX}(p-1) & c_{XX}(p-2) & \cdots & c_{XX}(0) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = -\begin{pmatrix} c_{XX}(1) \\ c_{XX}(2) \\ \vdots \\ c_{XX}(p) \end{pmatrix}.$$

If all roots of $\alpha(z)$ are lying within the unit circle one can show that the coefficient matrix of the equation system, which is a symmetric Toeplitz matrix, is positive definite.

Thus, assuming $c_{XX}(0), \ldots, c_{XX}(p)$ to be known the equation system can be uniquely solved for $a_1, a_2, \ldots, a_p$, e.g.

by means of the Levinson-Durbin algorithm, and the variance of the white noise can be subsequently determined by the Yule-Walker equations for $m = 0$, i.e.

$$\sigma_Z^2 = c_{XX}(0) + \sum_{n=1}^{p} a_n \, c_{XX}(n).$$

Furthermore, for $m > 0$ the Yule-Walker equations show that the covariance function $c_{XX}(m)$ satisfies exactly the same difference equation as $a(B)X_m = 0$.

Consequently, the general solution is of the form

$$c_{XX}(\tau) = C_1\lambda_1^{|\tau|} + C_2\lambda_2^{|\tau|} + \ldots + C_p\lambda_p^{|\tau|},$$

where $C_1, C_2, \ldots, C_p$ are constants determined by the initial conditions, i.e. $c_{XX}(0), c_{XX}(1), \ldots, c_{XX}(p-1)$.

Supplement: (parametric spectrum estimation)

$$C_{XX}(\Omega) = \left|H(\Omega)\right|^2 C_{ZZ}(\Omega) = \sigma_Z^2 \Big/ \left|1 + \sum_{n=1}^{p} a_n e^{-j\Omega n}\right|^2$$

$$\text{Suppose } \; c_{XX}(\tau) \approx \hat{c}_{XX}(\tau) \;\Rightarrow\; \begin{cases} a_n\big|_{c_{XX}(\tau)} \approx \hat{a}_n\big|_{\hat{c}_{XX}(\tau)} \quad n = 1,\ldots,p \\ \sigma_Z^2 \approx \hat{\sigma}_Z^2\big|_{\hat{c}_{XX}(\tau),\hat{a}_1,\ldots,\hat{a}_p} \end{cases}$$

$$C_{XX}(\Omega) \approx \hat{C}_{XX}(\Omega) = \left|\hat{H}(\Omega)\right|^2 \hat{C}_{ZZ}(\Omega) = \hat{\sigma}_Z^2 \Big/ \left|1 + \sum_{n=1}^{p} \hat{a}_n e^{-j\Omega n}\right|^2$$

Exercise 2.7-1:
Properties of an AR(1)-Process

Exercise 2.7-2:
Properties of an AR(2)-Process

2.7.3 Moving-Average (MA)-Processes

(X_t) is said to be a moving-average process of order q (denoted by MA(q)) if it can be expressed in the form

$$X_t = Z_t + b_1 Z_{t-1} + \ldots + b_q Z_{t-q} = \sum\nolimits_{n=0}^{q} b_n Z_{t-n} \quad \text{with} \quad b_0 = 1,$$

where $b_1,\ldots,b_q$ are constants and (Z_t) is white noise.

Using again the backward shift operator B we can write

$$X_t = \left(\sum_{n=0}^{q} b_n B^n \right) Z_t = b(B) Z_t \quad \text{and} \quad X_t = \sum_{n=0}^{q} h_n Z_{t-n} \quad \text{with} \quad h_n = b_n .$$

Since (X_t) is a linear combination of uncorrelated random variables its mean and variance are readily obtained using the results of Chapter 1.10.3, i.e.

$$\mu_X = \mu_Z \sum\nolimits_{n=0}^{q} b_n \quad \text{and} \quad \sigma_X^2 = \sigma_Z^2 \sum\nolimits_{n=0}^{q} b_n^2 .$$

Furthermore, (X_t) is always stationary (irrespective of the values of $b_1,\ldots,b_q$) and has the covariance function

$$c_{XX}(\tau) = \mathrm{E}(X_t X_{t-\tau}) = \mathrm{E}\left(\sum_{n=0}^{q} b_n Z_{t-n} \sum_{m=0}^{q} b_m Z_{t-\tau-m} \right)$$

$$= \sum_{n=0}^{q} \sum_{m=0}^{q} b_n b_m \mathrm{E}(Z_{t-n} Z_{t-\tau-m}) = \sum_{n=0}^{q} \sum_{m=0}^{q} b_n b_m c_{ZZ}(m+\tau-n)$$

$$= \sum_{n=0}^{q} \sum_{m=0}^{q} b_n b_m \sigma_Z^2 \delta_{m+\tau-n} = \begin{cases} \sigma_Z^2 \sum_{m=0}^{q-|\tau|} b_{m+|\tau|} b_m & |\tau| \le q \\ 0 & |\tau| > q \end{cases},$$

where $\mathrm{E}(Z_t) = 0 \Rightarrow \mathrm{E}(X_t) = 0$ has been exploited.

Supplement: $\sigma_Z^2 \sum_{n=0}^{q} \sum_{m=0}^{q} b_n b_m \delta_{m+\tau-n} = \sigma_Z^2 \sum_{m=0}^{q-|\tau|} b_{m+|\tau|} b_m$

$$\left(b_n b_m\right)_{\substack{n=0,\ldots,q\\ m=0,\ldots,q}} = \begin{pmatrix} b_0 b_0 & b_0 b_1 & b_0 b_2 & b_0 b_3 & \cdots & b_0 b_q \\ b_1 b_0 & b_1 b_1 & b_1 b_2 & b_1 b_3 & \cdots & b_1 b_q \\ b_2 b_0 & b_2 b_1 & b_2 b_2 & b_2 b_3 & \cdots & b_2 b_q \\ b_3 b_0 & b_3 b_1 & b_3 b_2 & b_3 b_3 & \cdots & b_3 b_q \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ b_q b_0 & b_q b_1 & b_q b_2 & b_q b_3 & \cdots & b_q b_q \end{pmatrix}$$

$\ldots, \tau = 2,\ \tau = 1,\ \tau = 0\ \ \tau = -1, \ldots$

For given covariances $c_{XX}(0),\ldots,c_{XX}(q)$ the parameters $b_1, b_2, \ldots, b_q$ and σ_Z^2 can be determined by solving the system of $(q+1)$ non-linear equations

$$c_{XX}(\tau) = \sigma_Z^2 \sum_{m=0}^{q-|\tau|} b_{m+|\tau|} b_m = \sum_{m=0}^{q-|\tau|} \gamma_{m+|\tau|} \gamma_m$$

with $\gamma_n = \sigma_Z b_n,\ \ \tau = 0,\ldots,q.$

Generally, such a non-linear equation system possesses 2^q solution vectors $(\gamma_0,\ldots,\gamma_q)^T$.

However, one can show that by imposing physically and system theoretically motivated constraints on the non-linear equation system the solution space can be reduce such that a unique solution can be derived.

<u>Exercise 2.7-3:</u>
Covariance function of an MA(q)-Process with equal weights

2.7.4 Auto-Regressive-Moving-Average (ARMA)-Processes

A process (X_t) that satisfies an equation of the form

$$X_t + a_1 X_{t-1} + \ldots + a_p X_{t-p} = Z_t + b_1 Z_{t-1} + \ldots + b_q Z_{t-q}$$

$$X_t + \sum_{n=1}^{p} a_n X_{t-n} = \sum_{n=0}^{q} b_n Z_{t-n} \quad \text{with} \quad b_0 = 1,$$

is called auto-regressive-moving-average process of order (p,q) (denoted by ARMA(p,q)).

Using the same operator notation introduced in the previous sections the ARMA(p,q) can be rewritten as

$$a(B)X_t = \left(1 + \sum_{n=1}^{p} a_n B^n\right) X_t = \left(\sum_{n=0}^{q} b_n B^n\right) Z_t = b(B) Z_t.$$

The solution of the equation can be represented by

$$X_t = \underbrace{A_1\lambda_1^t + A_2\lambda_2^t + \ldots + A_p\lambda_p^t}_{\substack{\text{solution of the homogeneous} \\ \text{equation } a(B)X_t=0}} + \underbrace{a^{-1}(B)b(B)Z_t}_{\substack{\text{particular solution of} \\ a(B)X_t=b(B)Z_t}},$$

where $\lambda_1,\lambda_2,\ldots,\lambda_p$ are the roots (assuming distinct roots) of the polynomial $\alpha(z)=a(z^{-1})z^p = z^p+\sum_{n=1}^{p} a_n z^{p-n}$.

Thus, asymptotic stationarity is provided if $|\lambda_n|<1$ for $n=1,\ldots,p$, i.e. the roots of $\alpha(z)$ must lie inside the unit circle.

Let (X_t) be stationary, then we may ignore the solution of $a(B)X_t = 0$ (which decays to zero), and the steady state solution can be written as

$$X_t = a^{-1}(B)b(B)Z_t = \left(\sum_{n=0}^{\infty} h_n B^n\right) Z_t = \sum_{n=0}^{\infty} h_n Z_{t-n}.$$

Now, multiplying both sides of the equation

$$X_t + \sum_{n=1}^{p} a_n X_{t-n} = \sum_{n=0}^{q} b_n Z_{t-n} \quad \left(\mathrm{E}(Z_t) = 0 \Rightarrow \mathrm{E}(X_t) = 0\right)$$

by X_{t-m} from the right and taking expectations we obtain

$$\mathrm{E}\left(\left(X_t + \sum_{n=1}^{p} a_n X_{t-n}\right) X_{t-m}\right) = \mathrm{E}\left(\left(\sum_{n=0}^{q} b_n Z_{t-n}\right) X_{t-m}\right)$$

$$\mathrm{E}(X_t X_{t-m}) + \sum_{n=1}^{p} a_n \mathrm{E}(X_{t-n} X_{t-m}) = \sum_{n=0}^{q} b_n \mathrm{E}(Z_{t-n} X_{t-m})$$

$$c_{XX}(m) + \sum_{n=1}^{p} a_n c_{XX}(m-n) = 0 \quad m > q.$$

This set of equations is sometimes called modified Yule-Walker equations.

For $m = q+1, q+2, ..., q+p$ the modified Yule-Walker equations can be expressed by

$$\begin{pmatrix} c_{XX}(q) & c_{XX}(q-1) & \cdots & c_{XX}(q-p+1) \\ c_{XX}(q+1) & c_{XX}(q) & \cdots & c_{XX}(q-p+2) \\ \vdots & \vdots & \ddots & \vdots \\ c_{XX}(q+p-1) & c_{XX}(q+p-2) & \cdots & c_{XX}(q) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{pmatrix} = -\begin{pmatrix} c_{XX}(q+1) \\ c_{XX}(q+2) \\ \vdots \\ c_{XX}(q+p) \end{pmatrix}.$$

The coefficient matrix of the equation system is obviously again a Toeplitz matrix but it is not anymore symmetric.

However, one can show that if all roots of $\alpha(z)$ are lying within the unit circle the coefficient matrix is regular.

Hence, assuming $c_{XX}(0), ..., c_{XX}(q+p)$ to be known the equation system can be uniquely solved for $a_1, a_2, ..., a_p$.

Subsequently, the parameters $a_1, a_2, \ldots, a_p$ and the covariances $c_{XX}(0), \ldots, c_{XX}(q+p)$ allow the calculation of the covariance function $c_{YY}(\tau)$ of the MA(q)-process

$$Y_t = b(B)Z_t = a(B)X_t = \sum_{n=0}^{p} a_n X_{t-n} \quad \text{with} \quad a_0 = 1$$

as follows.

$$c_{YY}(\tau) = \mathrm{E}\left(Y_t Y_{t-\tau}\right) = \mathrm{E}\left(\sum_{n=0}^{p} a_n X_{t-n} \sum_{m=0}^{p} a_m X_{t-\tau-m}\right)$$

$$= \sum_{n=0}^{p}\sum_{m=0}^{p} a_n a_m \mathrm{E}(X_{t-n} X_{t-\tau-m}) = \sum_{n=0}^{p}\sum_{m=0}^{p} a_n a_m c_{XX}(\tau + m - n)$$

$$= \sum_{k=-p}^{p}\sum_{m=0}^{p-|k|} a_{m+|k|} a_m c_{XX}(\tau - k)$$

Finally, after determining the covariances $c_{YY}(0),\dots,c_{YY}(q)$ the parameters $b_1,b_2,\dots,b_q$ and σ_Z^2 can be determined by solving the equation system

$$c_{YY}(\tau)=\sigma_Z^2\sum_{m=0}^{q-|\tau|}b_{m+|\tau|}b_m$$

$$=\sum_{m=0}^{q-|\tau|}\gamma_{m+|\tau|}\gamma_m \quad \text{with} \quad \gamma_n=\sigma_Z b_n \quad n=0,\dots,q.$$

which usually possesses 2^q solution vectors $(\gamma_0,\dots,\gamma_q)^T$.

As already mentioned in conjunction with MA(q)-Processes one can impose constraints on the non-linear equation system such that a unique solution can be derived.

References to Chapter 2

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