Underwater Acoustics and Sonar Signal Processing

Contents

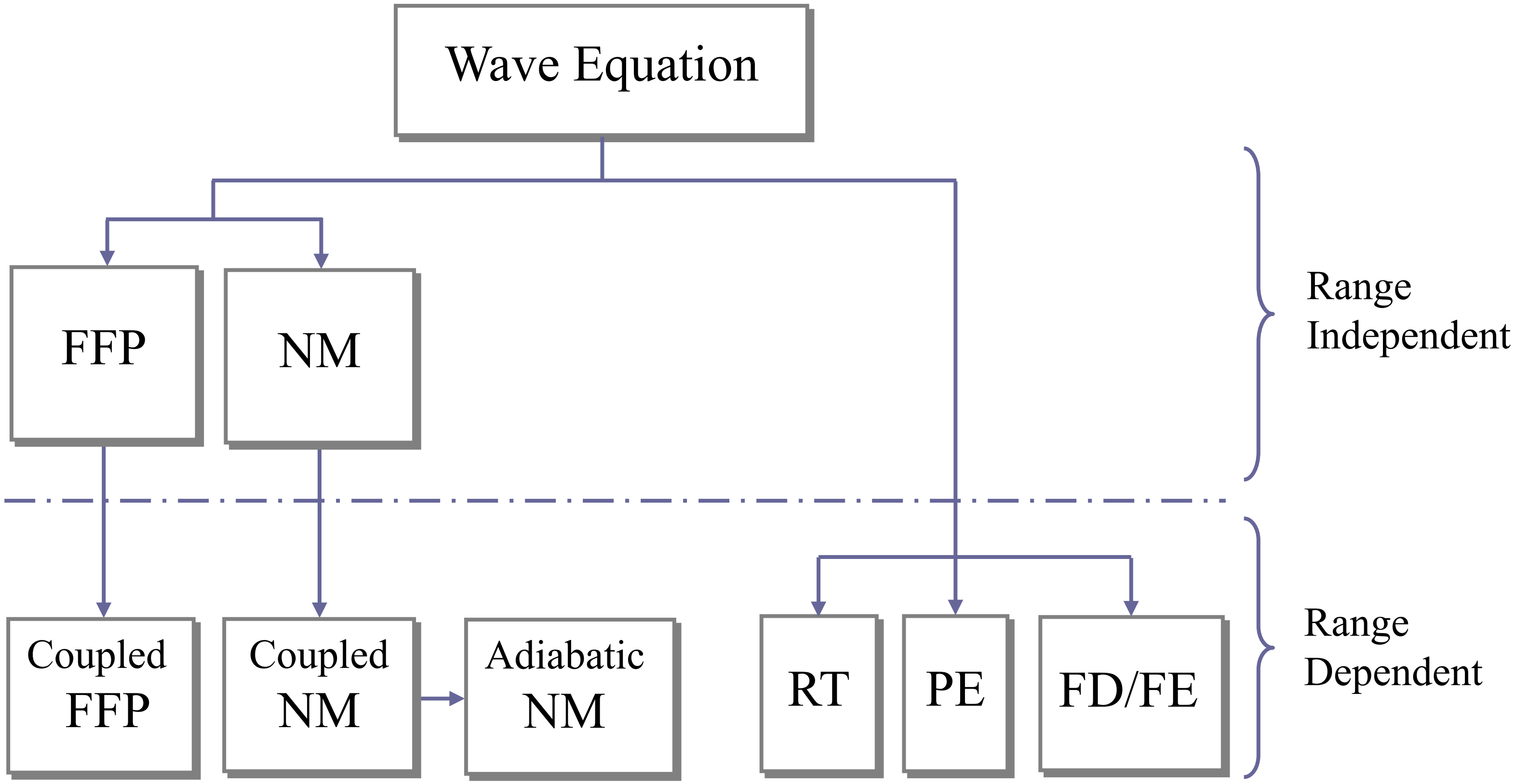
1 Fundamentals of Ocean Acoustics
2 Sound Propagation Modeling
3 Sonar Antenna Design
4 Sonar Signal Processing
5 Array Processing

2 Sound Propagation Modeling

Sound propagation in the ocean is mathematically formulated by the wave equation, whose parameters and boundary conditions are descriptive of the ocean environment. As summarized in the figure below, there are a variety of different techniques available for solving the wave equation (numerically) for evaluating sound propagation in the sea.

<u>Abbreviations</u>

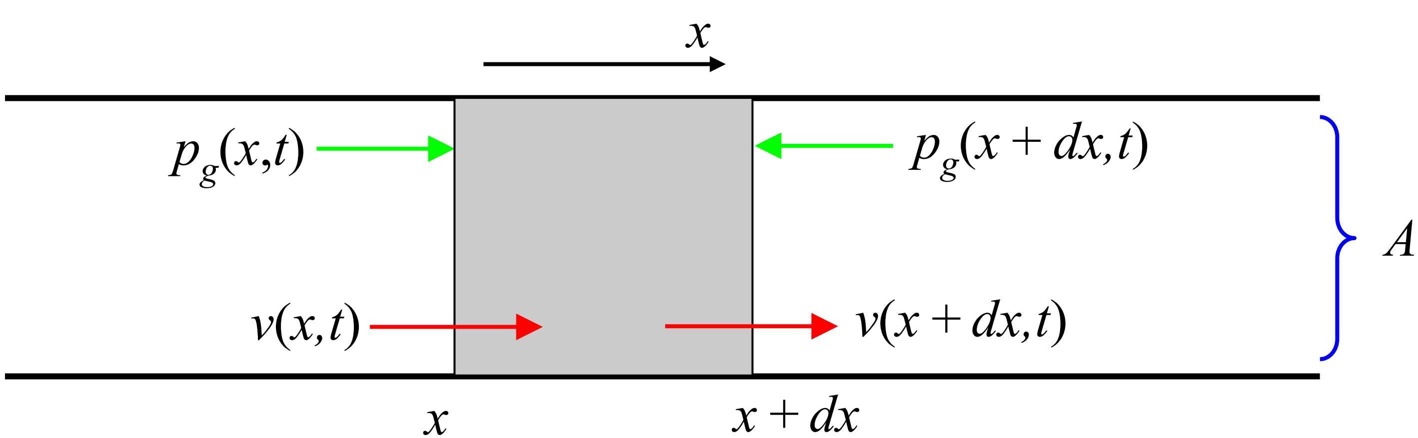
FE:	Finite Element	PE:	Parabolic Equation
FD:	Finite Difference	FFP:	Fast Field Program
NM:	Normal Mode	RT:	Ray Tracing


Wave Equation
FFP
NM
Range
Independent
Coupled
FFP
Coupled
NM
Adiabatic
NM
RT
PE
FD/FE
Range
Dependent

2.1 The Wave Equation

The wave equation in an ideal fluid can be derived from hydrodynamics and the adiabatic relation between pressure and density.

The following figure is used to derive the wave equation by exploiting the equation for conservation of mass, the Euler's equation and the adiabatic equation of state.



For deriving the following equations we define the total pressure and the total density as follows.

$$p_g = p_0 + p \quad \text{and} \quad \rho_g = \rho_0 + \rho,$$

where p_g, p_0, p, ρ_g, ρ_0 and ρ denote the total pressure, static pressure, change in pressure, total density, static density and change in density, respectively.

Continuity Equation

Employing the figure above the equation for the conservation of mass can be expressed by

$$\underbrace{\rho_g(x+dx,t)Av(x+dx,t) - \rho_g(x,t)Av(x,t)}_{\text{Resultant mass stream}} = \underbrace{-Adx \underbrace{\frac{\partial \rho_g}{\partial t}}_{\text{density variation}}}_{\text{Mass variation}}$$

and with

$$\frac{\rho_g(x+dx,t)v(x+dx,t)-\rho_g(x,t)v(x,t)}{dx}=\frac{\partial(\rho_g v)}{\partial x}$$

reformulated to obtain the so-called continuity equation

$$\frac{\partial(\rho_g v)}{\partial x}=-\frac{\partial\rho_g}{\partial t}=-\frac{\partial\rho}{\partial t}.$$

Euler's Equation

Using the figure above Newton's 2nd law can be written as

$$\underbrace{p_g(x,t)A-p_g(x+dx,t)A}_{\text{Total Force, }F}=\underbrace{\rho_g\underbrace{Adx}_{V}}_{m}\underbrace{\frac{dv}{dt}}_{a}$$

and by exploiting

$$dv = \frac{\partial v}{\partial t} dt + \frac{\partial v}{\partial x} dx, \text{ i.e. } \frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\frac{dx}{dt} = \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}$$

and

$$\frac{p_g(x,t) - p_g(x+dx,t)}{dx} = -\frac{p_g(x+dx,t) - p_g(x,t)}{dx} = -\frac{\partial p_g}{\partial x}$$

we obtain Euler's equation

$$-\frac{\partial p_g}{\partial x} = -\frac{\partial p}{\partial x} = \rho_g \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} \right).$$

Adiabatic Equation of State

$$p_g = p_0 \left(\frac{\rho_g}{\rho_0} \right)^{\kappa} = p_0 + \left. \frac{\partial p_g}{\partial \rho_g} \right|_{\rho_g = \rho_0} \rho + \frac{1}{2} \left. \frac{\partial^2 p_g}{\partial \rho_g^2} \right|_{\rho_g = \rho_0} \rho^2 + \ldots$$

where κ denotes the adiabatic exponent.

For convenience we define

$$c^2 = \left.\frac{\partial p_g}{\partial \rho_g}\right|_{\rho_g=\rho_0} = \kappa \frac{p_0}{\rho_0}\left(\frac{\rho_g}{\rho_0}\right)^{\kappa-1}\Bigg|_{\rho_g=\rho_0} = \kappa\frac{p_0}{\rho_0}$$

which turns out later to be the squared sound speed in an ideal fluid. For

$$p \ll p_0 \quad \text{and} \quad \rho \ll \rho_0$$

the adiabatic equation of state becomes approximately

$$p_g \cong p_0 + c^2\rho, \text{ i.e. } p \cong c^2\rho.$$

Since the time scale of oceanographic changes is much longer than the time scale of the acoustical propagation, we suppose that the material properties ρ_0 and c^2 are independent of time.

Taking the partial derivative of the continuity equation with respect to *t* and of Euler's equation with respect to *x* provides

$$\frac{\partial}{\partial t}\left(\frac{\partial(\rho_g v)}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial(\rho_g v)}{\partial t}\right) = -\frac{\partial^2 \rho}{\partial t^2}, \quad \rho = p/c^2$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \rho_g}{\partial t} v\right) + \frac{\partial}{\partial x}\left(\rho_g \frac{\partial v}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

and

$$-\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left(\rho_g \frac{\partial v}{\partial t}\right) + \frac{\partial}{\partial x}\left(\rho_g v \frac{\partial v}{\partial x}\right)$$

respectively. For lower particle velocities *v* as well as

$$p \ll p_0 \quad \text{and} \quad \rho \ll \rho_0$$

the terms

$$\frac{\partial}{\partial x}\left(\frac{\partial \rho_g}{\partial t} v\right) \quad \text{and} \quad \frac{\partial}{\partial x}\left(\rho_g v \frac{\partial v}{\partial x}\right)$$

can be neglected. Thus, the former equations simplify to

$$\frac{\partial}{\partial x}\left(\rho_g \frac{\partial v}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \quad \text{and} \quad -\frac{\partial^2 p}{\partial x^2} = \frac{\partial}{\partial x}\left(\rho_g \frac{\partial v}{\partial t}\right)$$

and provide by equating the 1dimensional linear wave equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}$$

which can be extended by straightforward argumentation to the 3-dimensional case given by

$$\Delta p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \quad \text{with} \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Helmholtz Equation

Suppose harmonic pressure oscillation, i.e.

$$p(x,y,z,t) = P(x,y,z)\exp(j\omega t),$$

we obtain

$$\Delta P + k^2 P = 0 \quad \text{with} \quad k = \omega/c = 2\pi/\lambda .$$

If P possesses spherical symmetry, i.e. P is only depending on R, the Laplacian in spherical coordinates simplifies to

$$\Delta = \frac{\partial^2}{\partial R^2} + \frac{2}{R}\frac{\partial}{\partial R}.$$

Hence, the spherical wave solution of the Helmholtz equation is

$$P(R) = \frac{A\exp(-jkR)}{R} \quad \text{with} \quad R = \sqrt{(x-x_S)^2 + (y-y_S)^2 + (z-z_S)^2},$$

where x_S, y_S and z_S are the coordinates of an omnidirectional point source (pulsating sphere of small radius).

Another simple and important solution is provided by the plane wave

$$P(x,y,z) = A\exp\left(-j(k_x x + k_y y + k_z z)\right),$$

where k_x, k_y and k_z are the wave numbers that satisfy

$$k^2 = \mathbf{k}^T\mathbf{k} = k_x^2 + k_y^2 + k_z^2, \quad k = \omega/c = 2\pi/\lambda.$$

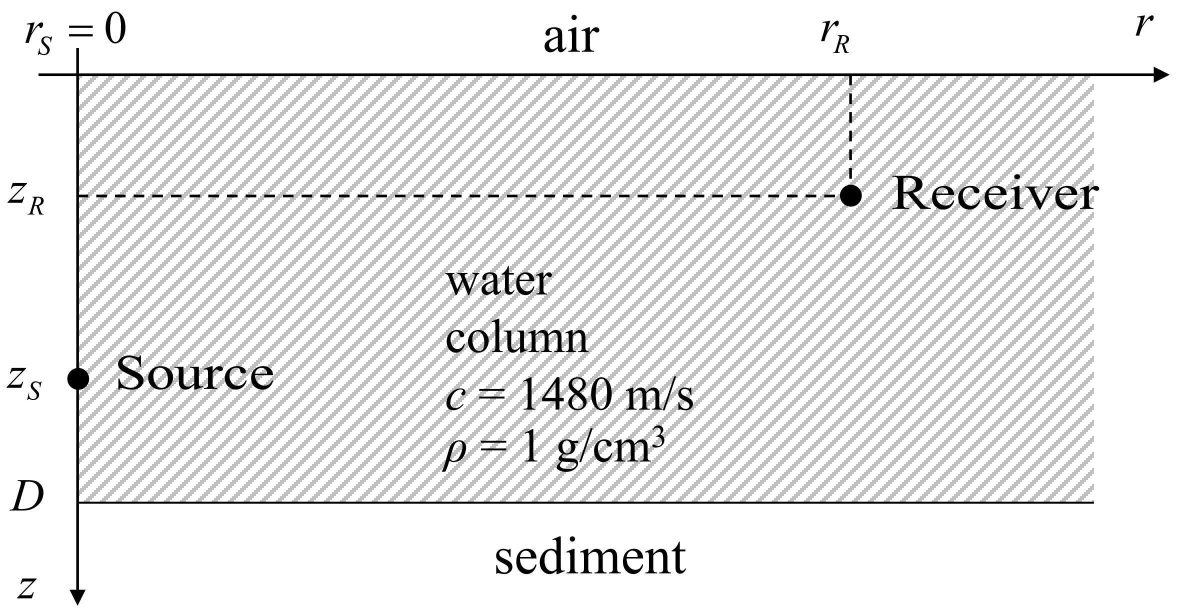
The wave vector $\mathbf{k}$ can also be expressed by

$$\mathbf{k} = (k_x, k_y, k_z)^T = k\left(\cos\varphi\cos\theta, \sin\varphi\cos\theta, \sin\theta\right)^T,$$

where φ and θ denote the azimuth and elevation, respectively.

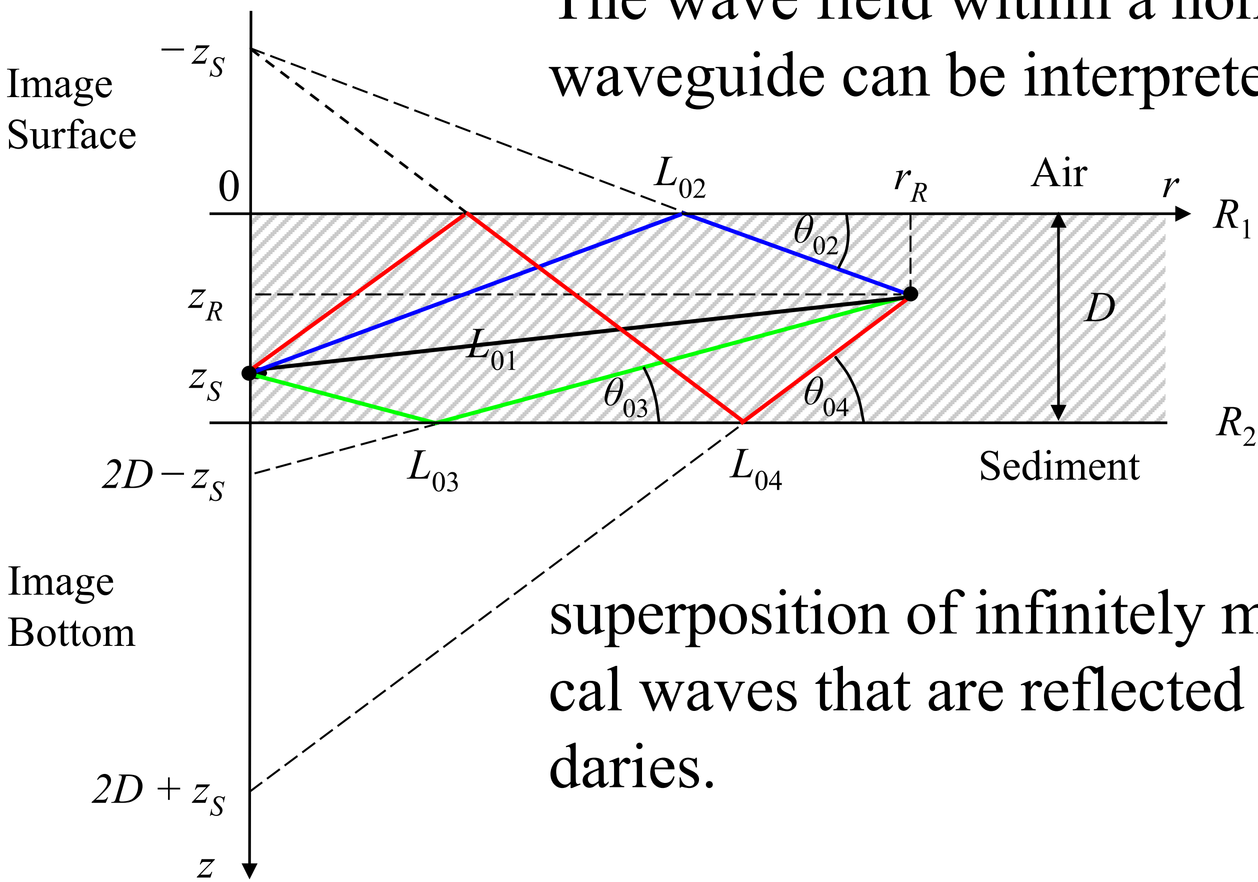
2.2 Homogeneous Waveguide

Suppose the medium within infinitely extended boundaries is homogeneous.



For the given point source coordinates $(0, z_S)$ the pressure shall be determined at an arbitrary receiver location (r_R, z_R).

2.2.1 Image Source Approach



The wave field within a homogeneous waveguide can be interpreted as the superposition of infinitely many spherical waves that are reflected at the boundaries.

As a first approximation, the sound pressure in the waveguide can be determined by superimposing the four contributions indicated in the figure above, i.e.

$$P(r_R, z_R, \omega) = A(\omega)\left(\frac{e^{-jkL_{01}}}{L_{01}} + R_1(\theta_{02},\omega)\frac{e^{-jkL_{02}}}{L_{02}} + \right.$$

$$\left. + R_2(\theta_{03},\omega)\frac{e^{-jkL_{03}}}{L_{03}} + R_1(\theta_{04},\omega)R_2(\theta_{04},\omega)\frac{e^{-jkL_{04}}}{L_{04}}\right)$$

with $L_{01} = \sqrt{r_R^2 + (z_R - z_S)^2}$, and

$L_{02} = \sqrt{r_R^2 + (z_S + z_R)^2}$, $\theta_{02} = \arctan\left((z_S + z_R)/r_R\right)$,

$L_{03} = \sqrt{r_R^2 + (2D - z_S - z_R)^2}$, $\theta_{03} = \arctan\left((2D - z_S - z_R)/r_R\right)$,

$L_{04} = \sqrt{r_R^2 + (2D + z_S - z_R)^2}$ $\theta_{04} = \arctan\left((2D + z_S - z_R)/r_R\right)$.

Continuation of the image source technique in multiples m = 1,2,… of groups of four contributions provides

$$P(r_R, z_R, \omega) = A(\omega) \sum_{m=0}^{\infty} \Bigg(R_1^m(\theta_{m1}, \omega) R_2^m(\theta_{m1}, \omega) \frac{e^{-jkL_{m1}}}{L_{m1}} + R_1^{m+1}(\theta_{m2}, \omega) R_2^m(\theta_{m2}, \omega) \frac{e^{-jkL_{m2}}}{L_{m2}} + R_1^m(\theta_{m3}, \omega) R_2^{m+1}(\theta_{m3}, \omega) \frac{e^{-jkL_{m3}}}{L_{m3}} + R_1^{m+1}(\theta_{m4}, \omega) R_2^{m+1}(\theta_{m4}, \omega) \frac{e^{-jkL_{m4}}}{L_{m4}} \Bigg)$$

with

$$L_{m1} = \sqrt{r_R^2 + (2Dm - z_S + z_R)^2}, \quad L_{m3} = \sqrt{r_R^2 + (2D(m+1) - z_S - z_R)^2},$$

$$L_{m2} = \sqrt{r_R^2 + (2Dm + z_S + z_R)^2}, \quad L_{m4} = \sqrt{r_R^2 + (2D(m+1) + z_S - z_R)^2}$$

and

$$\theta_{m1}=\arctan\left((2Dm-z_S+z_R)/r_R\right),\qquad \theta_{m3}=\arctan\left(\left(2D(m+1)-z_S-z_R\right)/r_R\right),$$

$$\theta_{m2}=\arctan\left((2Dm+z_S+z_R)/r_R\right),\qquad \theta_{m4}=\arctan\left(\left(2D(m+1)+z_S-z_R\right)/r_R\right).$$

Taking into account that the reflection coefficients at the ocean surface and bottom can be approximated by

$R \approx -1$ water-air-interface

$R \approx 1$ water-hard bottom-interface

the calculation of the sound pressure simplifies to

$$P(r,z,\omega)=A(\omega)\sum_{m=0}^{\infty}(-1)^m\left(\frac{\mathrm{e}^{-ikL_{m1}}}{L_{m1}}-\frac{\mathrm{e}^{-ikL_{m2}}}{L_{m2}}+\frac{\mathrm{e}^{-ikL_{m3}}}{L_{m3}}-\frac{\mathrm{e}^{-ikL_{m4}}}{L_{m4}}\right).$$

Assignment 6:

Develop a Matlab program for determining $P(r,z,\omega)$ for the following parameters.

Signal parameters

- Sinusoidal waveform
- Amplitude: $A = 1$,
- Frequency: $f = 10$ Hz, 100 Hz, 1 kHz and 10 kHz

Waveguide parameters

- Water depth: $D = 20$ m
- Source location: $r_S = 0$ m, $z_S = 5$ m
- Receiver location: $(r_R, z_R)^T \in [0,500] \times [0,D]$
- Surface/Bottom Reflection: $R_1 = -1$ (calm), $R_2 = 1$ (hard bottom)
- Sound speed: $c = 1480$ m/s

Depict the pressure distribution $P(r,z,\omega)$ in colour coded two dimensional diagrams and interpret the results.

2.2.2 Normal Mode Approach

For cylinder symmetric sound propagation, i.e. $p = p(r,z,t)$, the wave equation is given by

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2}.$$

Let us suppose that p can be expressed by $p \propto f(r)g(z)h(t)$. Hence, insertion in the wave equation provides

$$g(z)h(t)\frac{d^2 f}{dr^2} + \frac{g(z)h(t)}{r}\frac{df}{dr} + f(r)h(t)\frac{d^2 g}{dz^2} = \frac{f(r)g(z)}{c^2}\frac{d^2 h}{dt^2}$$

and after some manipulations

$$\frac{1}{f(r)}\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right) + \frac{1}{g(z)}\frac{d^2 g}{dz^2} = \frac{1}{c^2}\frac{1}{h(t)}\frac{d^2 h}{dt^2}.$$

For harmonic sources with $h(t) = A\,e^{j\omega t}$ we obtain

$$\underbrace{\frac{1}{f(r)}\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr}\right)}_{r\text{-dependent term}} + \underbrace{\frac{1}{g(z)}\frac{d^2 g}{dz^2}}_{z\text{-dependent term}} = -\frac{\omega^2}{c^2} = -k^2.$$

For all values of r and z, the r-dependent and z-dependent term are equal to constants. With the separation constant $-k_r^2$ for the radial and $-k_z^2$ for the vertical term, the separated ordinary differential equations are

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + k_r^2 f = 0 \quad \text{and} \quad \frac{d^2 g}{dz^2} + k_z^2 g = 0 \quad \text{with} \quad k^2 = k_r^2 + k_z^2.$$

The first equation is a zero-order Bessel equation. Its solution can be written in terms of a zero order Hankel function, i.e.

$$f(r) = \begin{cases} H_0^{(1)}(k_r r) = J_0(k_r r) + jY_0(k_r r) \\ H_0^{(2)}(k_r r) = J_0(k_r r) - jY_0(k_r r) \end{cases}$$

where

J_0 = zero-order Bessel function of the 1st kind, (besselj(…)),
Y_0 = zero-order Bessel function of the 2nd kind, (bessely(…)), also known as zero-order Neumann function N_0,
$H_0^{(1)}$ = zero-order Hankel function of the 1st kind, (besselh(…)),
$H_0^{(2)}$ = zero-order Hankel function of the 2nd kind, (besselh(…)),
both are also known as zero-order Bessel function of the 3rd kind.

The asymptotic form of the Hankel function for $k_r r \to \infty$ is

$$H_0^{(1)}(k_r r) \cong \sqrt{\frac{2}{\pi k_r r}}\, e^{j\left(k_r r - \frac{\pi}{4}\right)} \quad \text{and} \quad H_0^{(2)}(k_r r) \cong \sqrt{\frac{2}{\pi k_r r}}\, e^{-j\left(k_r r - \frac{\pi}{4}\right)},$$

where $H_0^{(1)}$ and $H_0^{(2)}$ represent the converging and diverging cylindrical wave, respectively.

The second equation represents an ordinary linear differential equation, where g has to satisfy the boundary conditions

$$\begin{aligned} g(0) &= 0 \qquad (R=-1) \\ g(D) &= \max \quad (R=1) \end{aligned}.$$

<u>Step I:</u> *(Elementary Solution)*

$$g(z) = e^{\lambda z} \Rightarrow g''(z) = \lambda^2 e^{\lambda z}$$

$$g''(z) + k_z^2 g(z) = 0 \Rightarrow \lambda^2 + k_z^2 = 0 \Rightarrow \lambda_{1,2} = \pm jk_z$$

With $\mathrm{Re}\{\exp(jk_z z)\} = \cos(k_z z)$ and $\mathrm{Im}\{\exp(jk_z z)\} = \sin(k_z z)$ the set of independent solutions is given by $\{\cos(k_z z), \sin(k_z z)\}$.

Thus, the general solution of the ordinary linear differential equation can be expressed by

$$g(z) = A\cos(k_z z) + B\sin(k_z z).$$

<u>Step II</u>: (Boundary Conditions)

$$g(0) = 0 \Rightarrow A = 0$$

$$g(D) = B\sin(k_z D) = \max \Rightarrow \left|\sin(k_z D)\right| = 1$$

$$\Rightarrow k_z D = (2m-1)\pi/2, \quad m = 1, 2, \ldots$$

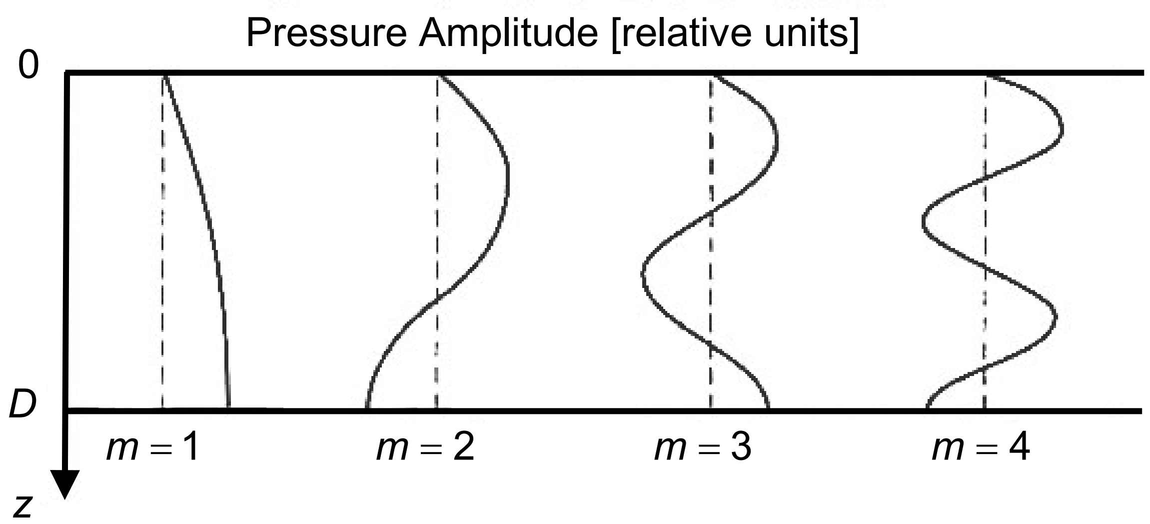
The boundary conditions are satisfied for a discrete set of values of k_z. Hence, we obtain

$$k_{z,m} = (2m-1)\frac{\pi}{2D} \qquad \text{(eigenvalues)}$$

and

$$g_m(z) = B_m \sin\left((2m-1)\frac{\pi}{2D} z \right) \qquad \text{(eigenfunctions)}$$

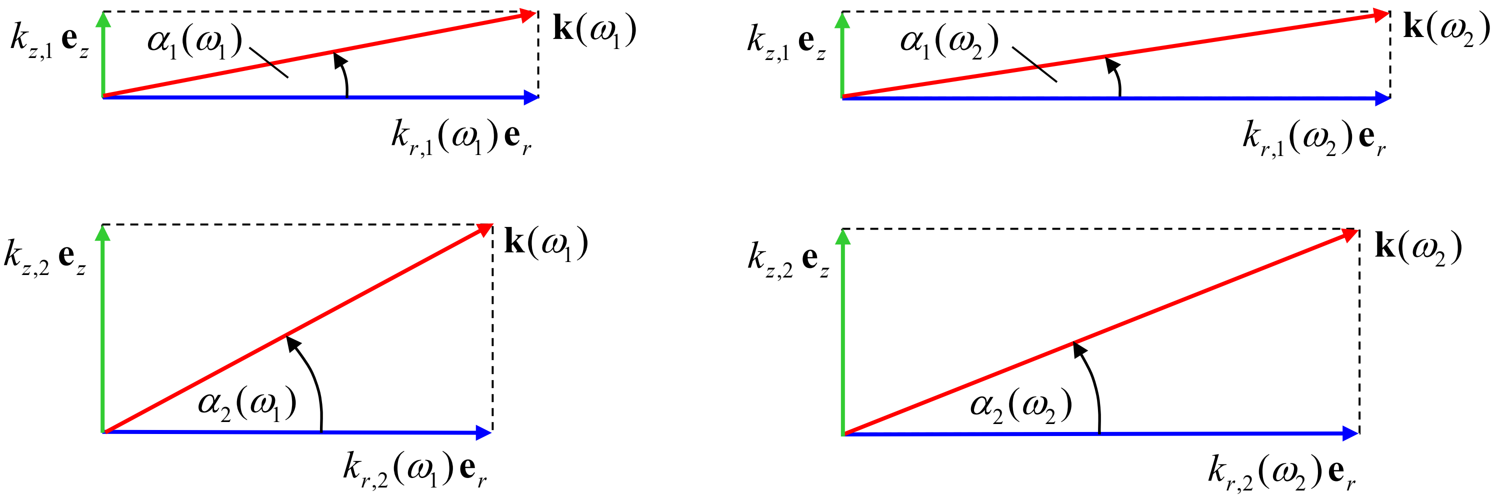
for $m = 1,2,\ldots$ The solutions are called modes because they describe the natural ways in which the system vibrates.



The eigenvalues $k_{r,m}^2$ and $k_{z,m}^2$ are related by $k^2 = k_{r,m}^2 + k_{z,m}^2$.

Considering $k_{r,m}^2$ and $k_{z,m}^2$ to be the horizontal and vertical component of **k**, respectively, we can write

$$k_{r,m}(\omega)=k(\omega)\cos\big(\alpha_m(\omega)\big) \quad \text{and} \quad k_{z,m}=k(\omega)\sin\big(\alpha_m(\omega)\big).$$



Consequently, we obtain $f_m(r)\cong\sqrt{\dfrac{2}{\pi k_{r,m} r}}\,e^{-j\left(k_{r,m} r-\frac{\pi}{4}\right)}$.

<u>Step III</u>: (Initial Conditions)

The last step is to add the fundamental solutions

$$g(z) = \sum_{m=1}^{\infty} B_m \sin(k_{z,m} z) = \sum_{m=1}^{\infty} B_m \sin\left((2m-1)\frac{\pi}{2D} z \right)$$

in such a way that the initial condition

$$g(z) = \phi(z)$$

is satisfied. Substituting the sum into the initial condition gives

$$\phi(z) = \sum_{m=1}^{\infty} B_m \sin(k_{z,m} z).$$

By multiplying each side of this equation with $\sin(k_{z,n} z)$ and integrating from 0 to D, we obtain

$$\int_0^D \phi(z) \sin(k_{z,n} z) dz = B_n \int_0^D \sin^2(k_{z,n} z) dz = B_n \frac{D}{2}$$

due to the orthogonality property

$$\int_0^D \sin(k_{z,m}z)\sin(k_{z,n}z)\,dz = \begin{cases} 0 & m \neq n \\ D/2 & m = n \end{cases}.$$

Hence, the B_n are determined by

$$B_n = \frac{2}{D}\int_0^D \phi(z)\sin(k_{z,n}z)\,dz \quad n = 1,2,\ldots$$

which are the Fourier coefficients of $\phi(z)$ ($\Rightarrow$ uniqueness)

For a source function given by

$$\phi(z) = \delta(z - z_S)$$

we have

$$B_n = \frac{2}{D}\int_0^D \delta(z - z_S)\sin(k_{z,n}z)dz = \frac{2}{D}\sin(k_{z,n}z_S)$$

and accordingly

$$g_m(z) = \frac{2}{D}\sin(k_{z,m} z_S)\sin(k_{z,m} z).$$

Finally, for the boundary and initial condition given above the solution of the wave equation can be expressed by

$$\begin{aligned} p(r,z,t) &= h(t)\sum_{m=1}^{\infty} g_m(z) f_m(r) \\ &= A e^{j\omega t}\sum_{m=1}^{\infty}\frac{2}{D}\sin(k_{z,m} z_S)\sin(k_{z,m} z)\sqrt{\frac{2}{\pi k_{r,m} r}}e^{-j\left(k_{r,m} r-\frac{\pi}{4}\right)} \\ &= \frac{A}{D}\sqrt{\frac{8}{\pi r}}e^{j\left(\omega t+\frac{\pi}{4}\right)}\sum_{m=1}^{\infty}\frac{\sin(k_{z,m} z_S)\sin(k_{z,m} z)e^{-jk_{r,m} r}}{\sqrt{k_{r,m}}}. \end{aligned}$$

2.3 Inhomogeneous Waveguide

If the sound speed in the water column is not constant the medium/waveguide is called inhomogeneous.

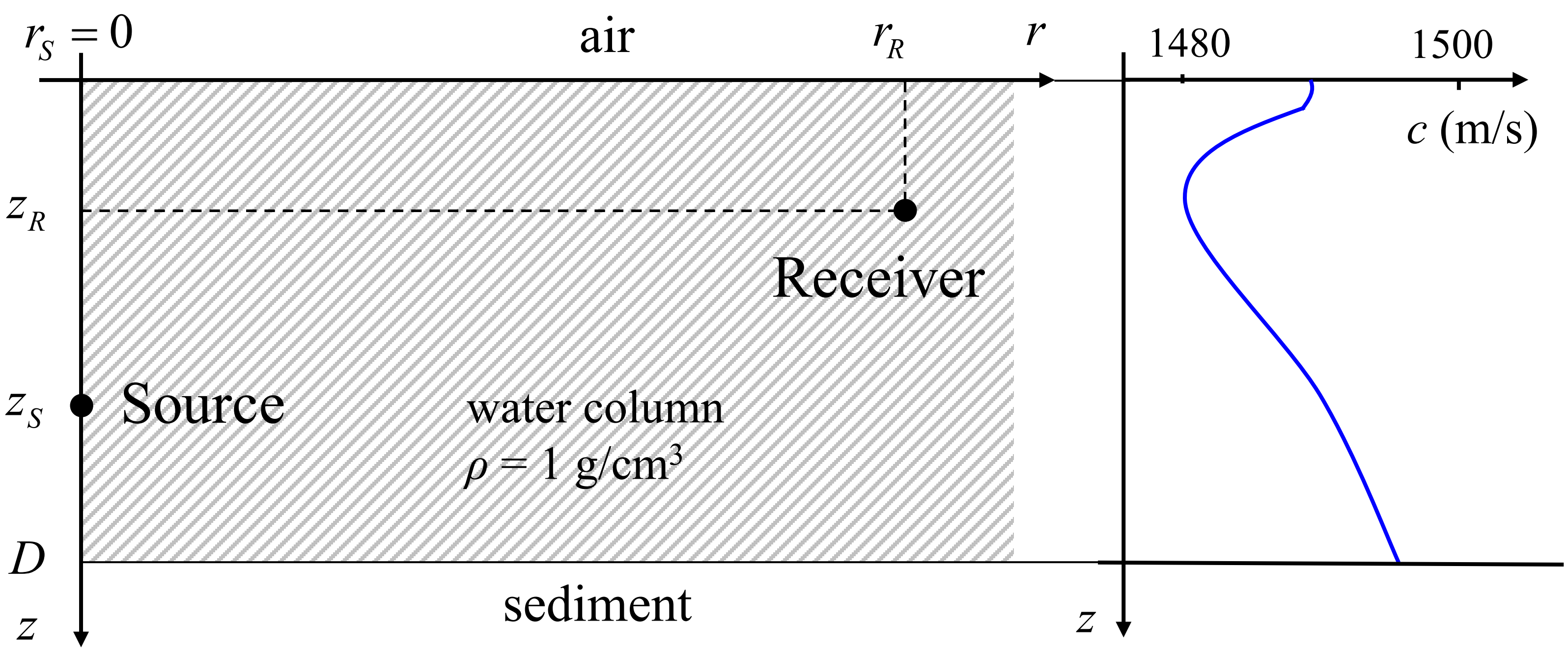
However, one generally assumes that the waveguide is cylinder symmetric with regard to the source location and is either

- range independent, i.e. the medium is horizontally stratified such that $c = c(z)$ is only a function of depth z

or

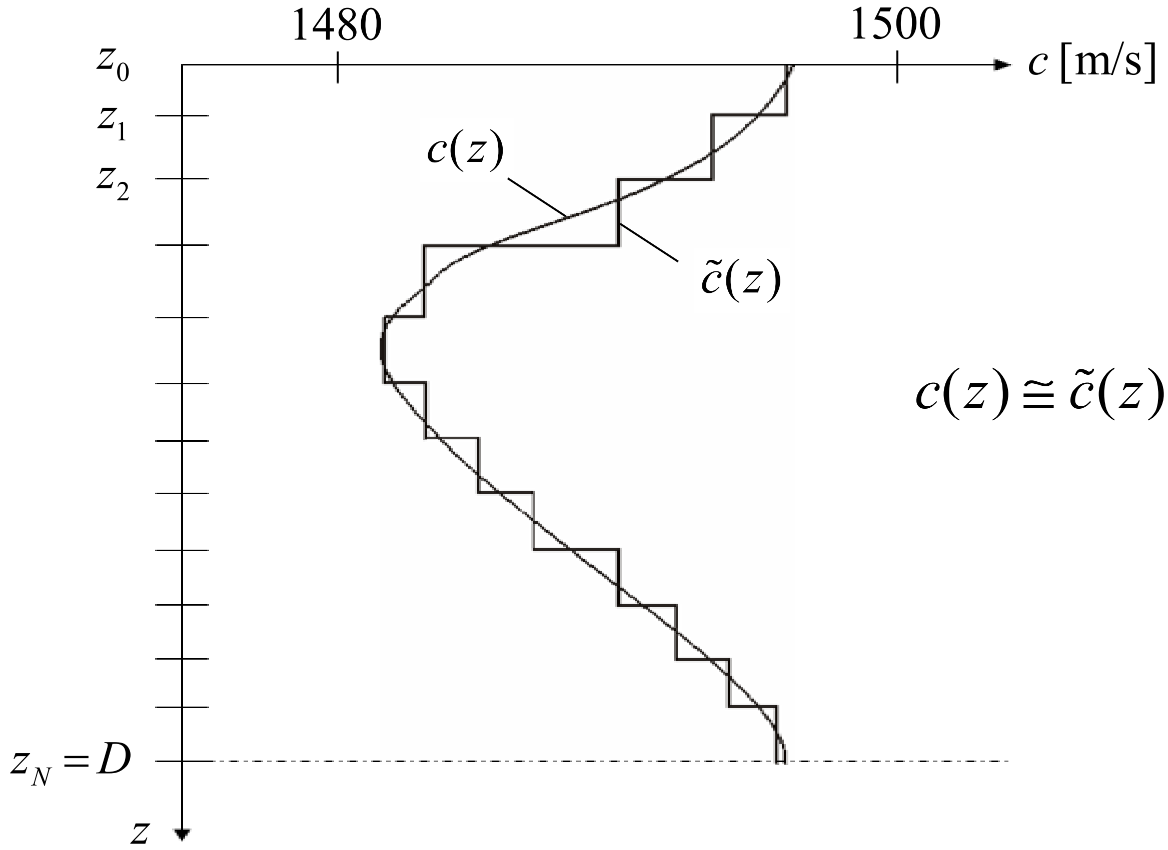
- range dependent, i.e. the sound speed varies versus horizontal range and depth such that $c = c(r,z)$ is a function of range r and depth z.

In the following range independent scenarios are assumed for simplicity.



2.3.1 Ray-Tracing, zero order sound speed approximation

The sound speed profile $c(z)$ is approximated by a staircase function.

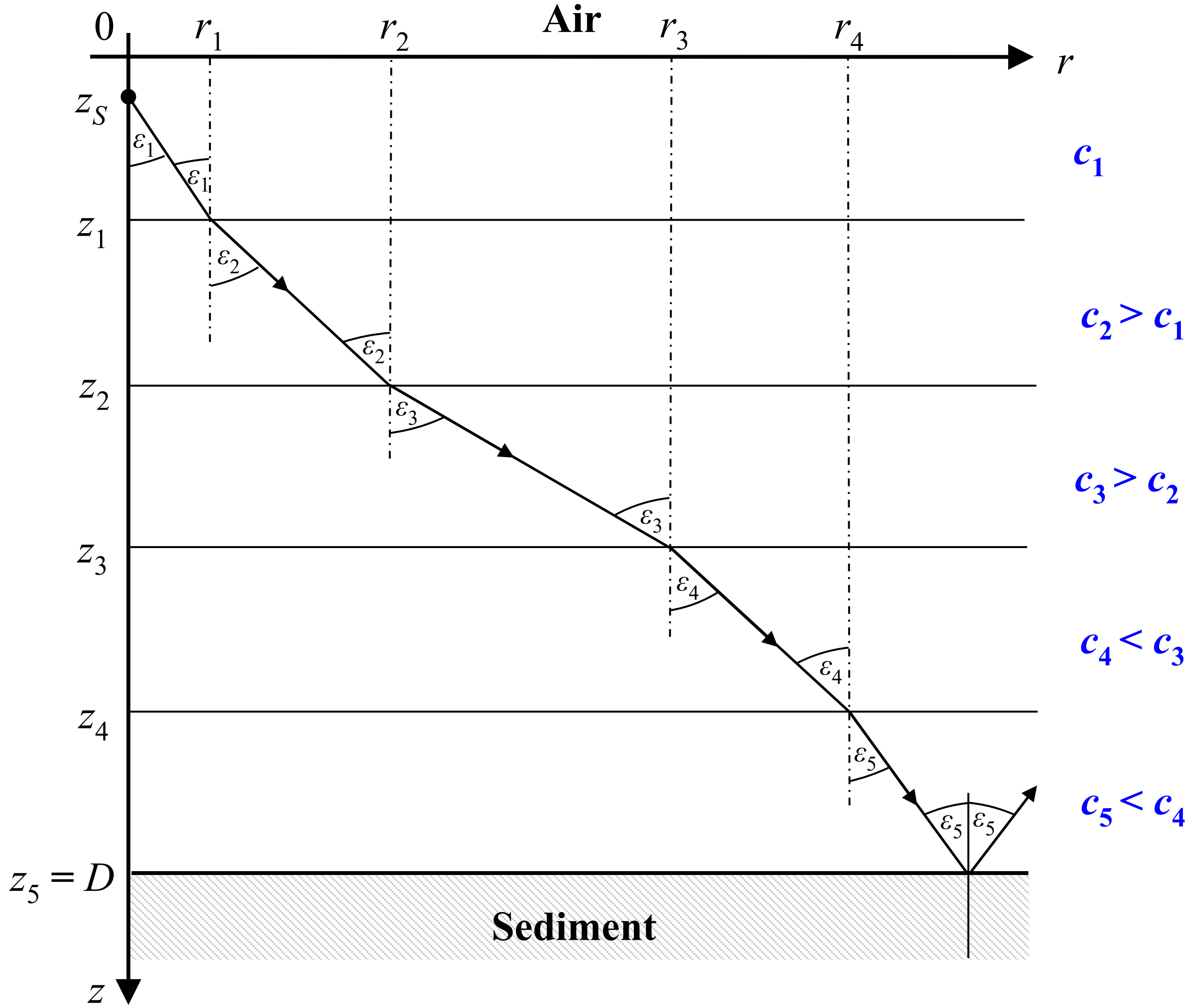


$$c(z) \cong \tilde{c}(z)$$

Snell's Law:

$$\frac{\sin \varepsilon_n}{\sin \varepsilon_{n+1}} = \frac{c_n}{c_{n+1}}$$

$$n = 1, \ldots, N$$



The ray trace can be determined by applying Snell's law at each boundary layer.

$$\frac{\sin\varepsilon_1}{c_1} = \frac{\sin\varepsilon_2}{c_2} = \ldots = \frac{\sin\varepsilon_N}{c_N} = a = \text{const.}$$

At the n-th boundary layer ($0 =$ surface, $\ldots$, $N =$ bottom) holds

$$\sin\varepsilon_n = a\,c_n.$$

With

$$\cos x = \sqrt{1-\sin^2 x} \quad \text{and} \quad \tan x = \frac{\sin x}{\cos x},$$

we obtain

$$\cos\varepsilon_n = \sqrt{1-a^2c_n^2} \quad \text{and} \quad \tan\varepsilon_n = \frac{a\,c_n}{\sqrt{1-a^2c_n^2}}$$

such that the horizontal position and the travel time of the ray at the l-th boundary layer can be determined by

$$r(l,a)=\sum_{n=1}^{l}(z_n-z_{n-1})\tan\varepsilon_n \quad \text{and} \quad T(l,a)=\sum_{n=1}^{l}\frac{z_n-z_{n-1}}{c_n\cos\varepsilon_n}$$

respectively, where $z_0 = z_S$, $z_N = D$ and $l = 1,\ldots,N$.

2.3.2 Ray-Tracing, first order sound speed approximation

For any $z\in[0,D]$ Snell's law

$$\sin\varepsilon(z_S)/c(z_S)=\sin\varepsilon(z)/c(z)=a$$

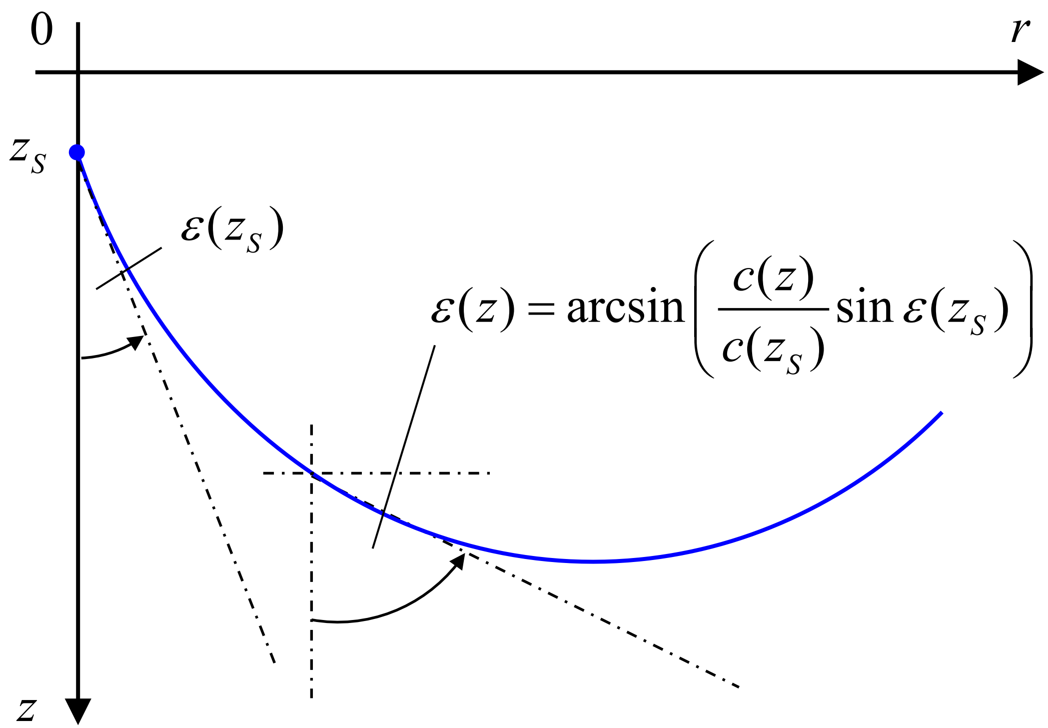
provides

$$\varepsilon(z)=\arcsin\left(\frac{c(z)}{c(z_S)}\sin\varepsilon(z_S)\right),$$

where

$$\sin\varepsilon(z) = a\,c(z) \quad \text{with} \quad a = \sin\varepsilon(z_S)/c(z_S)$$

has been exploited.



Using again the identities

$$\cos x = \sqrt{1-\sin^2 x} \quad \text{and} \quad \tan x = \sin x / \cos x\,,$$

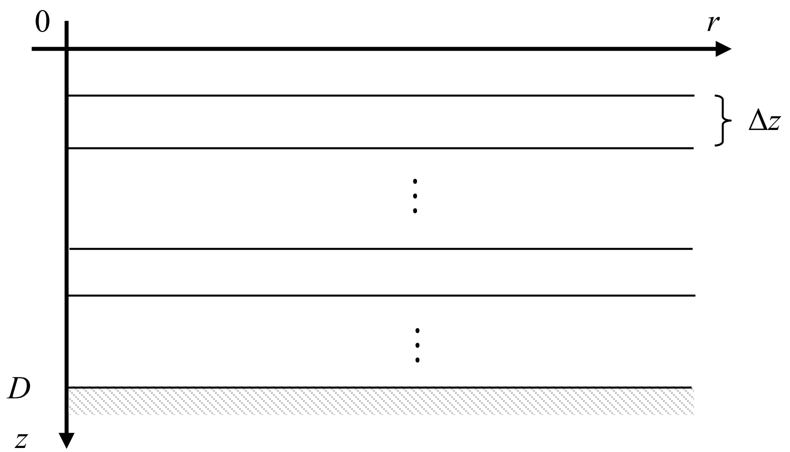
we obtain

$$\cos \varepsilon(z) = \sqrt{1-\sin^2 \varepsilon(z)} = \sqrt{1-a^2 c(z)^2}$$

and

$$\tan \varepsilon(z) = \sin \varepsilon(z) / \cos \varepsilon(z) = a\, c(z) \Big/ \sqrt{1-a^2 c(z)^2}\,.$$

Horizontal partitioning of the water column in thin layers of thickness Δz as indicated in the figure on the right side leads to



$$r(z,a) = r(z_S,a) + \sum_{n=1}^{N(z)} \Delta z \tan \varepsilon(z_S + n\Delta z).$$

For $\Delta z \to 0$ the Riemann sum becomes a Riemann integral so that we can write

$$r(z,a) = r(z_S,a) + \int_{z_S}^{z} \tan \varepsilon(z')\, dz'$$

$$= r(z_S,a) + \int_{z_S}^{z} \frac{ac(z')}{\sqrt{1-a^2c(z')^2}}\, dz'.$$

Correspondingly, the travel time

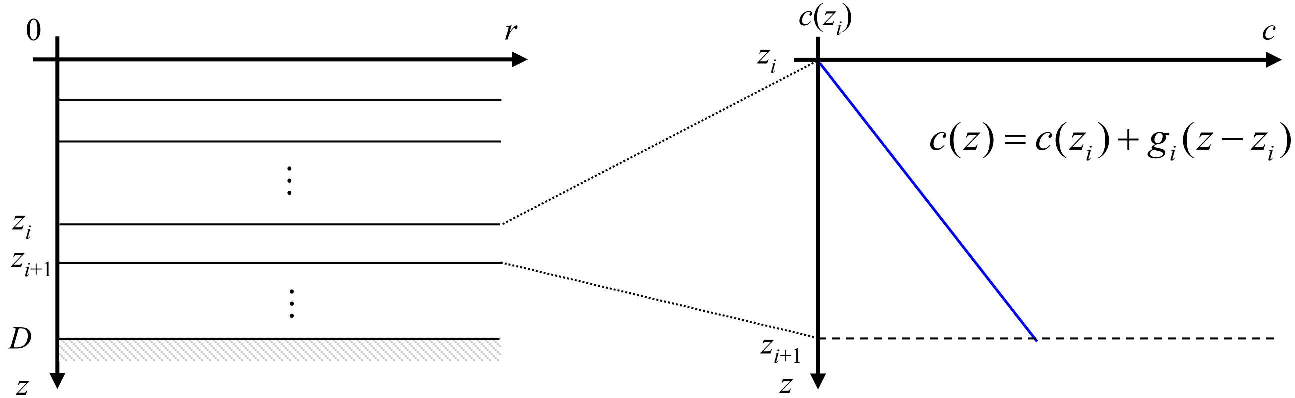
$$T(z,a) = T(z_S,a) + \sum_{n=1}^{N(z)} \frac{\Delta z}{c(z_S + n\Delta z)\cos \varepsilon(z_S + n\Delta z)}$$

results in

$$T(z,a) = T(z_S,a) + \int_{z_S}^{z} \frac{1}{c(z')\cos\varepsilon(z')}\,dz'$$

$$= T(z_S,a) + \int_{z_S}^{z} \frac{1}{c(z')\sqrt{1-a^2c(z')^2}}\,dz'$$

for $\Delta z \to 0$. Now, supposing the velocity profile can be approximated piecewise by linear functions, i.e.



the integrals

$$r(z,a) = r(z_i,a) + \int_{z_i}^{z} \frac{a\left(c(z_i)+g_i(z'-z_i)\right)}{\sqrt{1-a^2\left(c(z_i)+g_i(z'-z_i)\right)^2}}\,dz'$$

and

$$T(z,a) = T(z_i,a) + \int_{z_i}^{z} \frac{dz'}{\left(c(z_i)+g_i(z'-z_i)\right)\sqrt{1-a^2\left(c(z_i)+g_i(z'-z_i)\right)^2}}$$

can be solved for $z \in [z_i, z_{i+1}]$ analytically.

Substitution $v = c(z_i) + g_i(z-z_i)$ with $dv = g_i\,dz$ leads to

$$r(z,a) = r(z_i,a) + \int_{c(z_i)}^{c(z_i)+g_i(z-z_i)} \frac{a\,v}{\sqrt{1-a^2v^2}}\frac{1}{g_i}\,dv =$$

$$= r(z_i,a) - \sqrt{1-a^2v^2}\,\Big/ a g_i \Big|_{c(z_i)}^{c(z_i)+g_i(z-z_i)}$$

$$= r(z_i,a) + \left\{ \sqrt{1-a^2c(z_i)^2} - \sqrt{1-a^2\left(c(z_i)+g_i(z-z_i)\right)^2} \right\} \Big/ a g_i$$

The subsequent reformulation shows that rays follow circular paths in case of linear depth dependent velocity profiles.

$$r(z,a) - \left(r(z_i,a) + \frac{\sqrt{1-a^2c(z_i)^2}}{a g_i} \right) = -\frac{\sqrt{1-a^2\left(c(z_i)+g_i(z-z_i)\right)^2}}{a g_i}$$

$$\left(r(z,a) - \left(r(z_i,a) + \frac{\sqrt{1-a^2c(z_i)^2}}{a g_i} \right) \right)^2 = \frac{1-a^2\left(c(z_i)+g_i(z-z_i)\right)^2}{a^2 g_i^2}$$

$$\left(r(z,a)-\left(r(z_i,a)+\frac{\sqrt{1-a^2c(z_i)^2}}{ag_i}\right)\right)^2+\frac{\left(c(z_i)+g_i(z-z_i)\right)^2}{g_i^2}=\frac{1}{a^2g_i^2}$$

$$\left(r(z,a)-\left(r(z_i,a)+\frac{\sqrt{1-a^2c(z_i)^2}}{ag_i}\right)\right)^2+\left(z-\left(z_i-\frac{c(z_i)}{g_i}\right)\right)^2=\frac{1}{a^2g_i^2}$$

Moreover, the expression for the travel time can be similarly derived by

$$T(z,a)=T(z_i,a)+\int_{c(z_i)}^{c(z_i)+g_i(z-z_i)}\frac{1}{v\sqrt{1-a^2v^2}}\frac{1}{g_i}dv$$

$$=T(z_i,a)-\frac{1}{g_i}\ln\left(\frac{1+\sqrt{1-a^2v^2}}{av}\right)\Bigg|_{c(z_i)}^{c(z_i)+g(z-z_i)}=$$

$$= T(z_i,a) + \frac{1}{g_i}\left\{ \ln\left(\frac{1+\sqrt{1-a^2c(z_i)^2}}{a\,c(z_i)}\right) - \ln\left(\frac{1+\sqrt{1-a^2\left(c(z_i)+g(z'-z_i)\right)^2}}{a\left(c(z_i)+g(z-z_i)\right)}\right)\right\}$$

$$= T(z_i,a)+\frac{1}{g_i}\ln\left(\frac{a\left(c(z_i)+g_i(z-z_i)\right)\left(1+\sqrt{1-a^2c(z_i)^2}\right)}{\left(1+\sqrt{1-a^2\left(c(z_i)^2+g_i(z-z_i)\right)^2}\right)a\,c(z_i)}\right).$$

Since explicit expressions $r(z,a)$, $T(z,a)$ are available for linear velocity profiles, computationally efficient and satisfactory accurate ray-path calculations can be carried out after piecewise linear approximation of the actual velocity profile.

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