Underwater Acoustics and Sonar Signal Processing

Contents

3 Sonar Antenna Design

The characteristic features of a sonar transmitting/receiving antenna are mainly determined by the

- geometry and shading of the antenna aperture
- properties of the individual transducers.

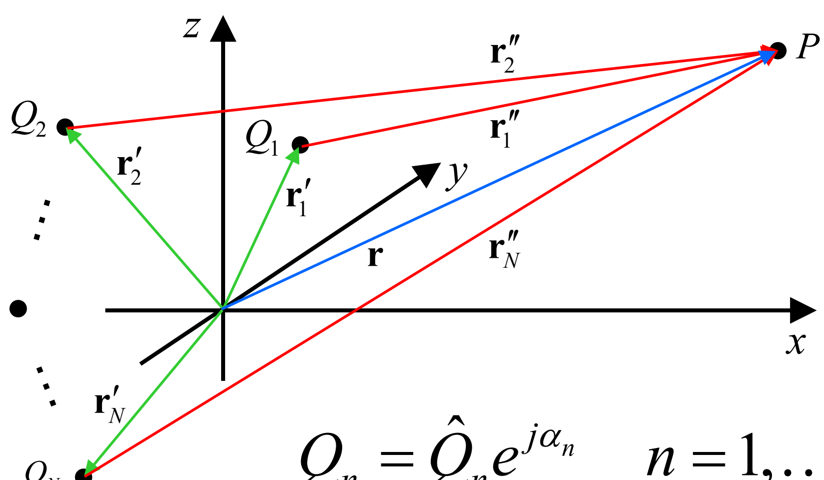
3.1 Design of the Antenna Aperture

3.1.1 Basic principles

Without loss of generality only transmitting antennas are considered in the sequel.
The aperture of an extended sound source can be understood as an arrangement of finite or infinite many point sources.

Finite or countable infinite many point sources

Let Q_n denote the amplitude of the n-th point source with $n = 1,\ldots,N$. The amplitude of the sound generated by the array of point sources at a particular point P in the three dimensional space is given by



$$Q_n = \hat{Q}_n e^{j\alpha_n} \qquad n = 1,\ldots,N$$

$$p(\mathbf{r},t) = \sum_{n=1}^{N} Q_n \frac{e^{j(\omega t - k|\mathbf{r}-\mathbf{r}'_n|)}}{|\mathbf{r}-\mathbf{r}'_n|} = \sum_{n=1}^{N} Q_n \frac{e^{j(\omega t - k|\mathbf{r}''_n|)}}{|\mathbf{r}''_n|} = \sum_{n=1}^{N} Q_n \frac{e^{j(\omega t - k r''_n)}}{r''_n}$$

where

$$r_n'' = |\mathbf{r}_n''| = |\mathbf{r} - \mathbf{r}_n'| \quad \text{with} \quad \mathbf{r}_n'' = \mathbf{r} - \mathbf{r}_n', \; \mathbf{r} = (x, y, z)^T, \; \mathbf{r}_n' = (x_n', y_n', z_n')^T$$

can be determined either by

$$r_n'' = \sqrt{(x - x_n')^2 + (y - y_n')^2 + (z - z_n')^2}$$

or with the law of cosine by

$$r_n'' = \sqrt{r^2 + r_n'^2 - 2\, r\, r_n' \cos\left(\measuredangle(\mathbf{r}, \mathbf{r}_n')\right)}$$

with

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}, \quad r_n' = |\mathbf{r}_n'| = \sqrt{x_n'^2 + y_n'^2 + z_n'^2}$$

and

$$\cos\left(\measuredangle(\mathbf{r}, \mathbf{r}_n')\right) = \frac{\mathbf{r}^T \mathbf{r}_n'}{r\, r_n'} = \frac{x\, x_n' + y\, y_n' + z\, z_n'}{\sqrt{x^2 + y^2 + z^2}\sqrt{x_n'^2 + y_n'^2 + z_n'^2}}.$$

Assuming, that $|\mathbf{r}| = r \gg r'_n = |\mathbf{r}'_n|$ for all $n = 1,\ldots,N$ (far-field), i.e. the vectors $\mathbf{r}''_n$ are nearly parallel, we obtain approximately

$$r''_n = r\sqrt{1 + (r'_n/r)^2 - 2(r'_n/r)\cos\left(\measuredangle(\mathbf{r},\mathbf{r}'_n)\right)}$$
$$\approx r\sqrt{1 - 2(r'_n/r)\cos\left(\measuredangle(\mathbf{r},\mathbf{r}'_n)\right)}$$

and with

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \ldots$$

finally

$$r''_n \approx r\left(1 - (r'_n/r)\cos\left(\measuredangle(\mathbf{r},\mathbf{r}'_n)\right)\right) = r - r'_n\cos\left(\measuredangle(\mathbf{r},\mathbf{r}'_n)\right).$$

Substituting r''_n in the exponent of $p(\mathbf{r},t)$ by

$$r_n'' \approx r - r_n' \cos\left(\measuredangle(\mathbf{r}, \mathbf{r}_n')\right)$$

and in the denominator by

$$r_n'' \approx r \quad \left(r \gg r_n'\right)$$

leads to

$$p(\mathbf{r}, t) = \frac{e^{j(\omega t - kr)}}{r} \sum_{n=1}^{N} Q_n e^{j k r_n' \cos(\measuredangle(\mathbf{r}, \mathbf{r}_n'))}.$$

After normalization by

$$\hat{Q} = \sum_{n=1}^{N} \hat{Q}_n, \text{ where } Q_n = \hat{Q}_n e^{j\alpha_n} \text{ for } n = 1, \ldots, N,$$

we obtain

$$p(\mathbf{r}, t) = \frac{\hat{Q}}{r} \tilde{b}(\varphi, \theta)\, e^{j(\omega t - kr)},$$

where

$$\tilde{b}(\varphi,\theta) = \frac{1}{\hat{Q}} \sum_{n=1}^{N} Q_n e^{j k r_n' \cos(\measuredangle(\mathbf{r},\mathbf{r}_n'))}$$

denotes the complex beam pattern with

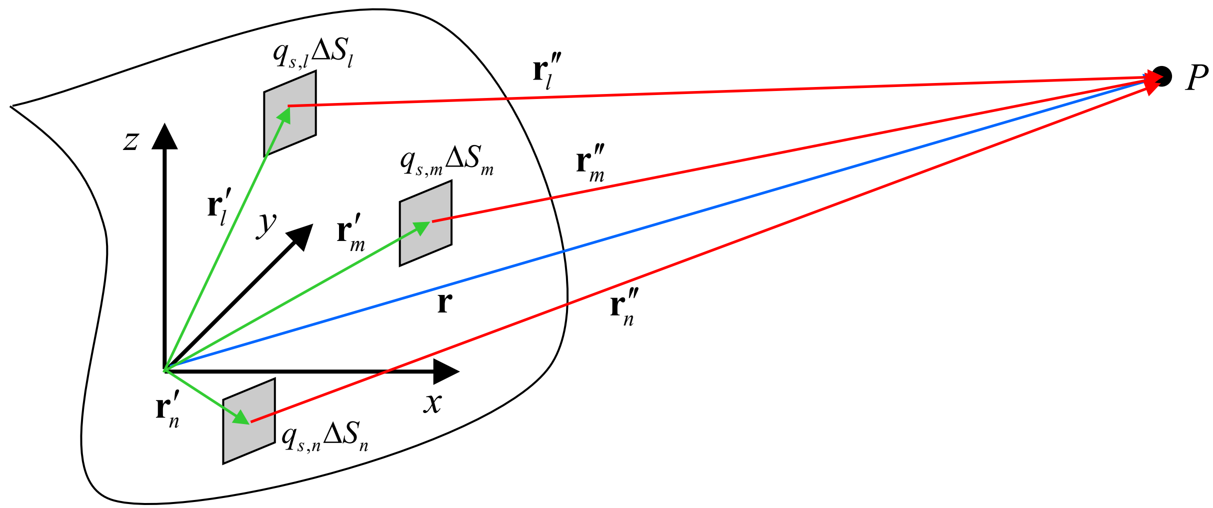
$$\mathbf{r} = r \begin{pmatrix} \cos\varphi\cos\theta \\ \sin\varphi\cos\theta \\ \sin\theta \end{pmatrix}, \quad \varphi - \text{Azimuth}, \;\; \theta - \text{Elevation}.$$

The squared magnitude in dB

$$\tilde{B}(\varphi,\theta) = 10\log_{10}\left|\tilde{b}(\varphi,\theta)\right|^2 = 20\log_{10}\left|\tilde{b}(\varphi,\theta)\right|$$

is called beam pattern.

Continuous apertures



The amplitude at a particular point P in the 3-dimensional space is given by

$$p(\mathbf{r},t) = \sum_k q_{s,k} \Delta S_k \frac{e^{j(\omega t - k|\mathbf{r}-\mathbf{r}_n'|)}}{|\mathbf{r}-\mathbf{r}_n'|} = \sum_k q_{s,k} \Delta S_k \frac{e^{j(\omega t - k r_n'')}}{r_n''},$$

where $Q_k = q_{s,k}\Delta S_k$ represents the amplitude of the k-th subaperture. For the far-field, i.e. $|\mathbf{r}| = r \gg r_n' = |\mathbf{r}_n'|\ \ \forall k$, the expression simplifies to

$$p(\mathbf{r},t) = \frac{e^{j(\omega t - kr)}}{r} \sum_k q_{s,k} e^{j k r_k' \cos(\measuredangle(\mathbf{r},\mathbf{r}_k'))} \Delta S_k .$$

Now, taking the limit $\max\{\Delta S_k\} \to 0$, the sum turns over into a surface integral

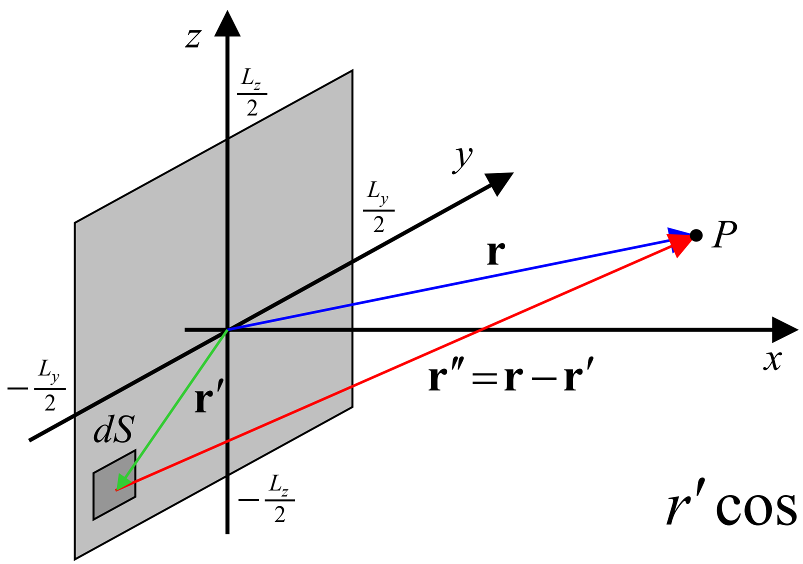
$$p(\mathbf{r},t) = \frac{e^{j(\omega t - kr)}}{r} \int_S q_s(\mathbf{r}') e^{j k r' \cos(\measuredangle(\mathbf{r},\mathbf{r}'))} dS,$$

where $q_s dS$ can be understood as the amplitude of a transmitter of infinitesimal aperture dS.

3.1.2 Continuous Apertures

Rectangular Aperture

For a 2-dimensional continuous aperture of rectangular shape and an coordinate system defined as depicted below,



$$\mathbf{r} = (x, y, z)^T$$

$$\mathbf{r}' = (0, y', z')^T$$

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r' \cos\left(\measuredangle(\mathbf{r}, \mathbf{r}')\right) = \cancel{r'} \frac{\mathbf{r}^T \mathbf{r}'}{r \cancel{r'}} = \frac{y y' + z z'}{r}$$

we obtain

$$p(\mathbf{r},t)=\frac{e^{j(\omega t-kr)}}{r}\int_{-L_z/2}^{L_z/2}\int_{-L_y/2}^{L_y/2} q_s(y',z')\,e^{jk\left(\frac{y}{r}y'+\frac{z}{r}z'\right)}dy'dz'.$$

Using the wave number vector representation given by

$$\mathbf{k}=\begin{pmatrix}k_x\\k_y\\k_z\end{pmatrix}=k\begin{pmatrix}x/r\\y/r\\z/r\end{pmatrix}=k\begin{pmatrix}\cos\varphi\cos\theta\\\sin\varphi\cos\theta\\\sin\theta\end{pmatrix}\quad\text{with}\quad k=\frac{2\pi}{\lambda}$$

the pressure can be expressed in spherical coordinates by

$$\tilde{p}(r,\varphi,\theta,t)=\frac{e^{j(\omega t-kr)}}{r}\int_{-L_z/2}^{L_z/2}\int_{-L_y/2}^{L_y/2} q_s(y',z')\,e^{jk(\sin\varphi\cos\theta\, y'+\sin\theta\, z')}dy'dz'.$$

After introducing

$$\hat{Q} = \int_{-L_z/2}^{L_z/2} \int_{-L_y/2}^{L_y/2} \hat{q}_s(y',z')\, dy'dz' \quad \text{with} \quad q_s(y',z') = \hat{q}_s(y',z')\, e^{j\alpha(y',z')},$$

we can define the complex beam pattern

$$\tilde{b}(\varphi,\theta) = \frac{1}{\hat{Q}} \int_{-L_z/2}^{L_z/2} \int_{-L_y/2}^{L_y/2} q_s(y',z')\, e^{jk(\sin\varphi\cos\theta\, y' + \sin\theta\, z')} dy'dz'$$

so that the pressure can be represented by

$$\tilde{p}(r,\varphi,\theta,t) = \tilde{b}(\varphi,\theta)\, \frac{\hat{Q}}{r}\, e^{j(\omega t - kr)}.$$

Moreover, the complex beam pattern can be interpreted as 2-dimensional Fourier Transform from the spatial (normalized aperture function) into the wave-number domain

$$b(k_y,k_z)=\frac{1}{\hat{Q}}\int_{-L_z/2}^{L_z/2}\int_{-L_y/2}^{L_y/2} q_s(y',z')\,e^{j(k_y y'+k_z z')}\,dy'dz'.$$

Assuming $q_s(y,z)$ to be constant over the entire rectangular aperture the complex beam pattern becomes

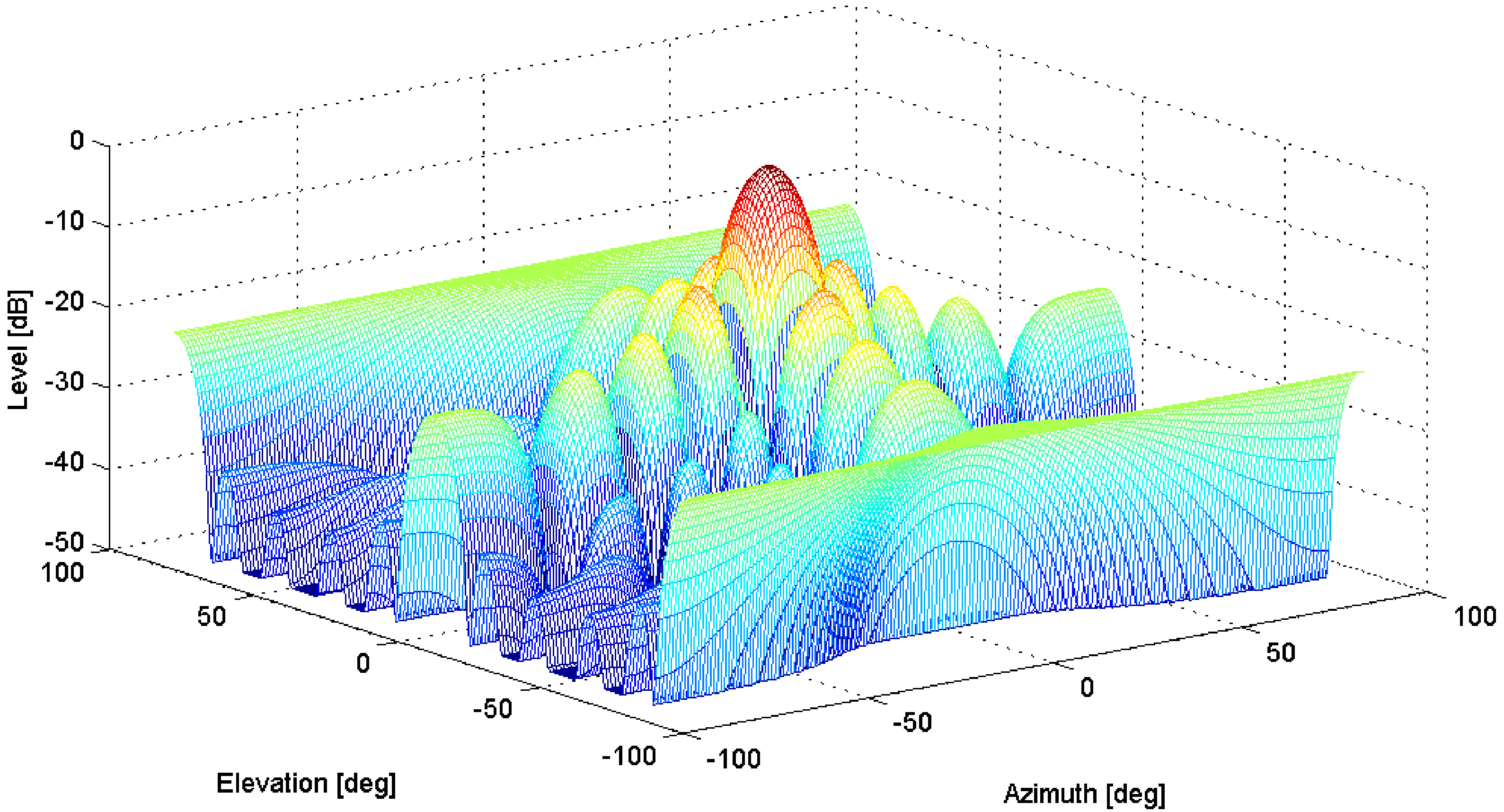
$$b(k_y,k_z)=\frac{1}{L_y}\int_{-L_y/2}^{L_y/2} e^{jk_y y'}dy'\cdot\frac{1}{L_z}\int_{-L_z/2}^{L_z/2} e^{jk_z z'}dz'=\frac{\sin(k_y L_y/2)}{k_y L_y/2}\cdot\frac{\sin(k_z L_z/2)}{k_z L_z/2}$$

or as function of azimuth φ and elevation θ finally

$$\tilde{b}(\varphi,\theta)=\frac{\sin(k\sin\varphi\cos\theta\,L_y/2)}{k\sin\varphi\cos\theta\,L_y/2}\cdot\frac{\sin(k\sin\theta\,L_z/2)}{k\sin\theta\,L_z/2}$$

$$=\frac{\sin(\pi\sin\varphi\cos\theta\,L_y/\lambda)}{\pi\sin\varphi\cos\theta\,L_y/\lambda}\cdot\frac{\sin(\pi\sin\theta\,L_z/\lambda)}{\pi\sin\theta\,L_z/\lambda}.$$

Typical beam pattern $\tilde{B}(\varphi,\theta)$ of a quadratic aperture with $q_s(y,z) = \text{const.}$

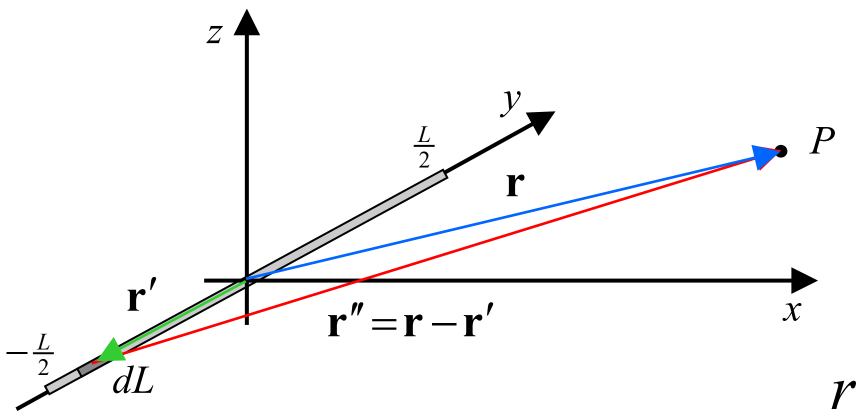


Linear Aperture

The aperture function of a continuous line shaped aperture can be expressed by

$$q_s(y,z) = q_l(y)\,\delta(z).$$

Choosing the coordinate system as depicted below



$$\mathbf{r} = (x,y,z)^T,\ \ \mathbf{r}' = (0,y',0)^T$$

$$r = |\mathbf{r}| = \sqrt{x^2+y^2+z^2}$$

$$r'\cos\big(\measuredangle(\mathbf{r},\mathbf{r}')\big) = r'\frac{\mathbf{r}^T\mathbf{r}'}{r\,r'} = \frac{y\,y'}{r}$$

the pressure becomes

$$p(\mathbf{r},t) = \frac{e^{j(\omega t - kr)}}{r} \int_{-L/2}^{L/2} q_l(y')\, e^{jk\frac{y}{r}y'}\, dy'.$$

Moreover, exploiting $y/r = \sin\varphi\cos\theta$ the pressure can be written in spherical coordinates as

$$\tilde{p}(r,\varphi,\theta,t) = \frac{e^{j(\omega t - kr)}}{r} \int_{-L/2}^{L/2} q_l(y')\, e^{jk\sin\varphi\cos\theta\, y'}\, dy'.$$

After employing the complex beam pattern

$$\tilde{b}(\varphi,\theta) = \frac{1}{\hat{Q}} \int_{-L/2}^{L/2} q_l(y')\, e^{jk\sin\varphi\cos\theta\, y'}\, dy',$$

we obtain

$$\tilde{p}(r,\varphi,\theta,t) = \tilde{b}(\varphi,\theta)\, \frac{\hat{Q}}{r}\, e^{j(\omega t - kr)},$$

where

$$\hat{Q} = \int_{-L/2}^{L/2} \hat{q}_l(y')\,dy' \quad \text{with} \quad q_l(y') = \hat{q}_l(y')\,e^{j\alpha(y')}.$$

The complex beam pattern can be again interpreted as Fourier Transform from the spatial into the wave-number domain

$$b(k_y) = \frac{1}{\hat{Q}} \int_{-L/2}^{L/2} q_l(y')\,e^{jk_y y'}\,dy'$$

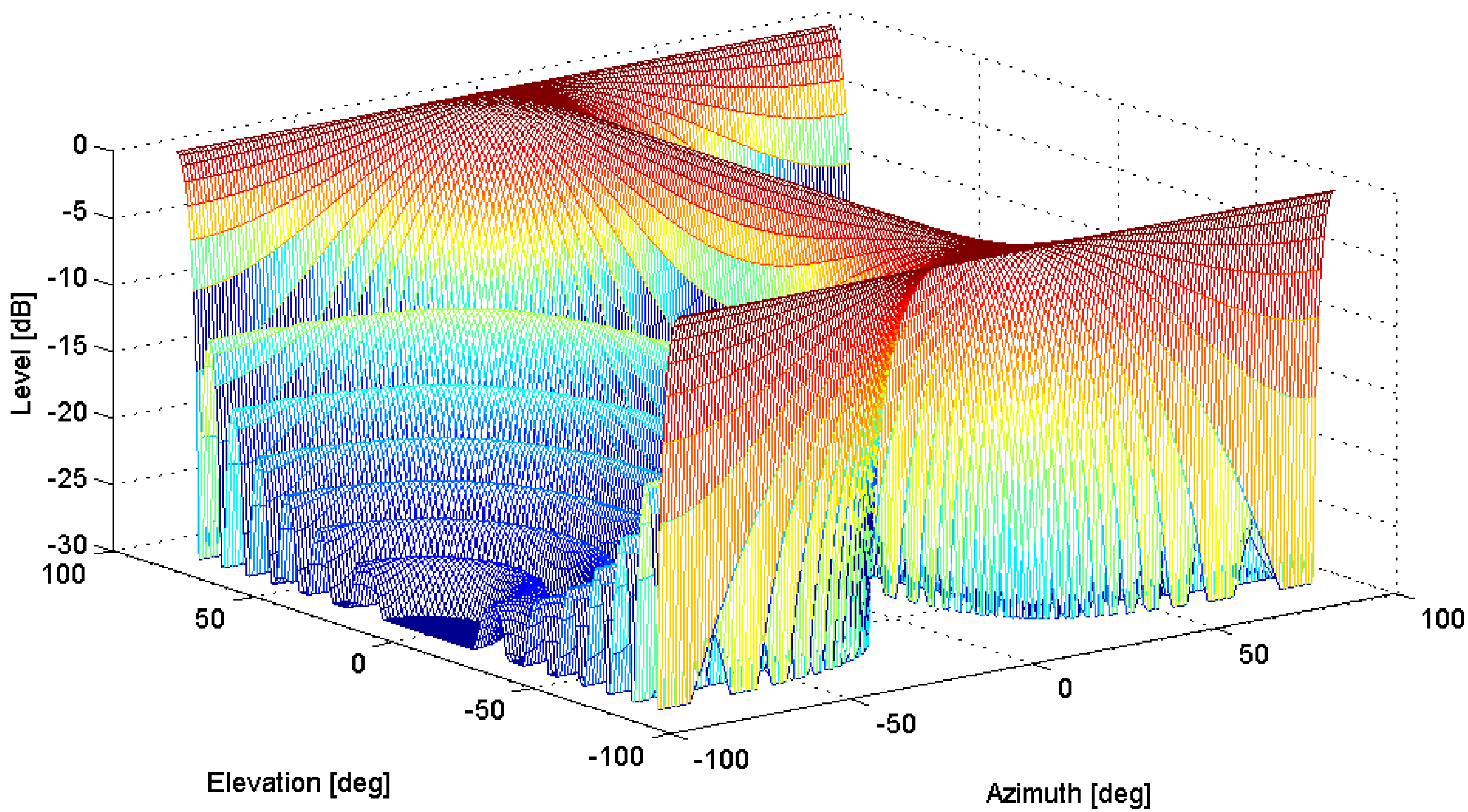
which for $q_l(y) = \text{const.}$ simplifies to

$$b(k_y) = \frac{1}{L} \int_{-L/2}^{L/2} e^{jk_y y'}\,dy' = \frac{\sin(k_y L/2)}{k_y L/2}$$

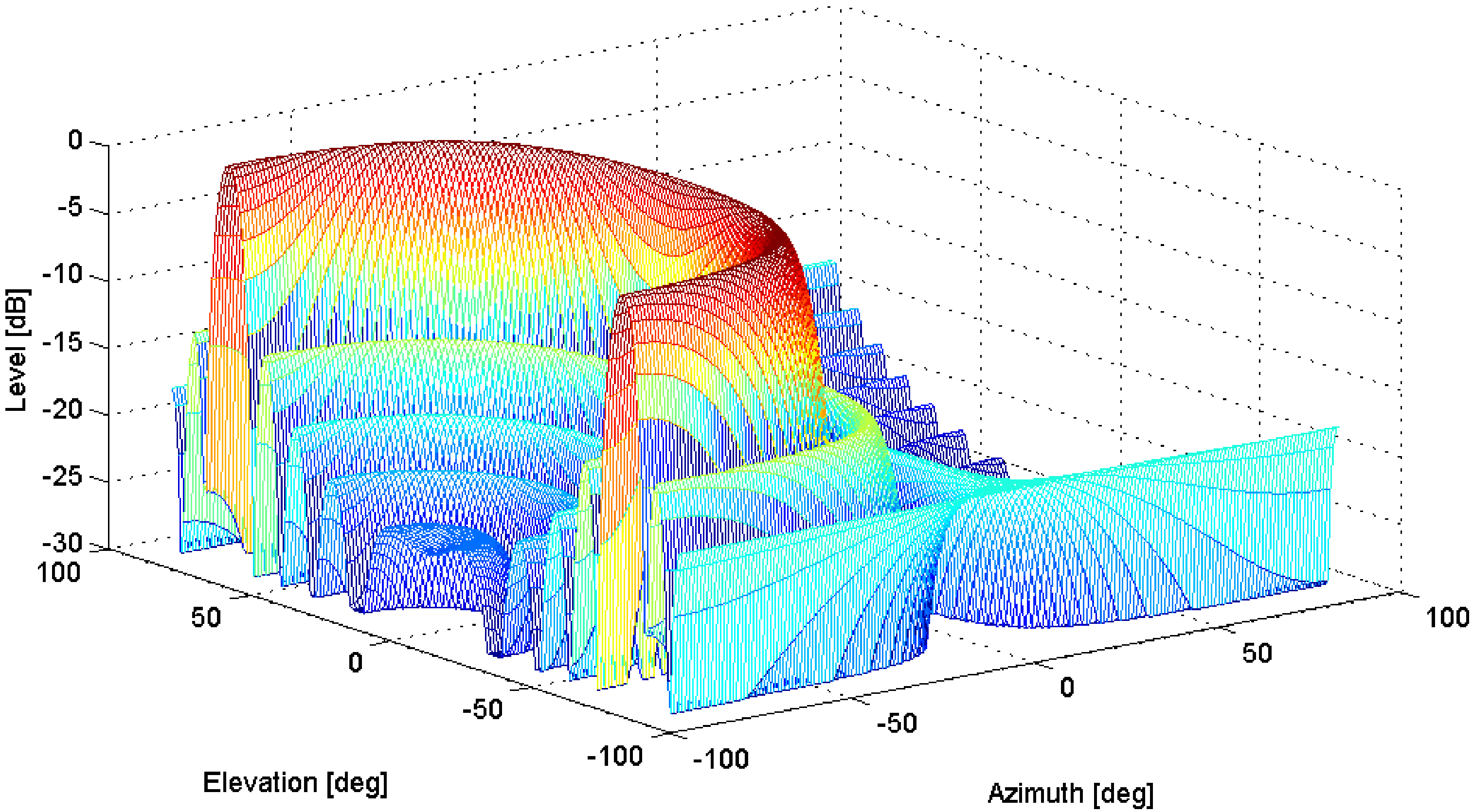
or as function of azimuth φ and elevation θ to

$$\tilde{b}(\varphi,\theta) = \frac{\sin(k \sin\varphi \cos\theta\ L/2)}{k \sin\varphi \cos\theta\ L/2} = \frac{\sin(\pi \sin\varphi \cos\theta\ L/\lambda)}{\pi \sin\varphi \cos\theta\ L/\lambda}.$$

Typical beam pattern $\tilde{B}(\varphi,\theta)$ of a linear aperture with $q_l(y) = \text{const.}$

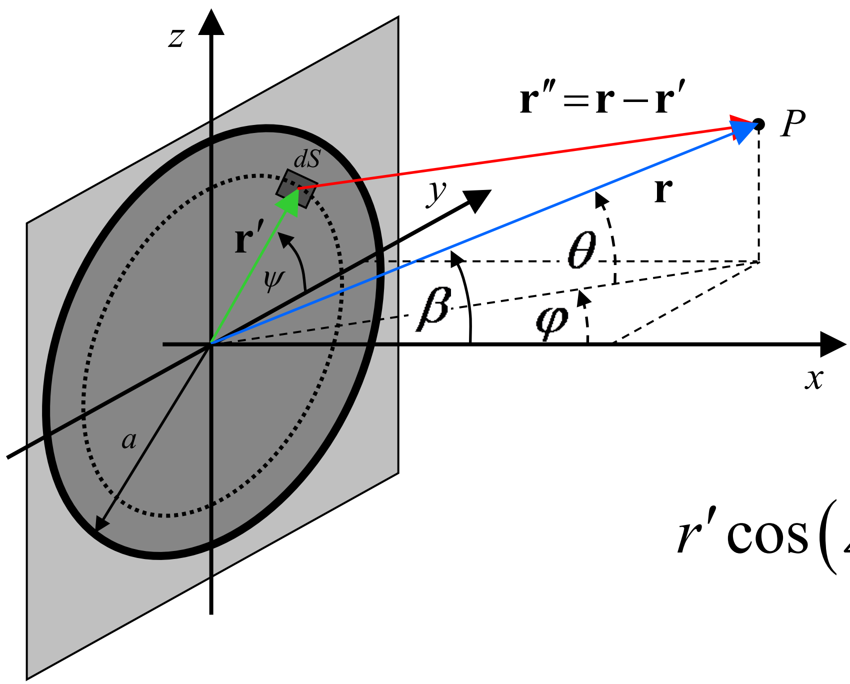


Typical beam pattern $\tilde{B}(\varphi,\theta)$ of a linear aperture with $q_l(y) = \exp(jk\sin(\pi/9)y)$



Circular Aperture

Now, we consider a circular disc shaped continuous aperture. With an coordinate system as introduced below,



$$\mathbf{r} = (x, y, z)^T$$

$$\mathbf{r}' = (0, y', z')^T$$

$$r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$r' \cos\left(\measuredangle(\mathbf{r}, \mathbf{r}')\right) = \cancel{r'} \frac{\mathbf{r}^T \mathbf{r}'}{r \cancel{r'}} = \frac{yy' + zz'}{r}$$

we can derive for the pressure the expression

$$p(\mathbf{r},t)=\frac{e^{j(\omega t-kr)}}{r}\int_{-a}^{a}\int_{-\sqrt{a^2-z'^2}}^{\sqrt{a^2-z'^2}} q_s(\mathbf{r}')\,e^{jk\left(\frac{y}{r}y'+\frac{z}{r}z'\right)}dy'dz'$$

which by employing

$$y/r=\sin\varphi\cos\theta \quad \text{and} \quad z/r=\sin\theta$$

can be reformulated in spherical coordinates to

$$\tilde{p}(r,\varphi,\theta,t)=\frac{e^{j(\omega t-kr)}}{r}\int_{-a}^{a}\int_{-\sqrt{a^2-z'^2}}^{\sqrt{a^2-z'^2}} q_s(\mathbf{r}')\,e^{jk(\sin\varphi\cos\theta\, y'+\sin\theta\, z')}dy'dz'.$$

To simplify the evaluation of the integral the following sub-stitution (polar coordinates) is introduced.

$$y'=r'\cos\psi,\;\; z'=r'\sin\psi \;\;\text{and}\;\; dy'dz'=\det(\mathbf{J})d\psi dr'=r'd\psi dr'$$

with

$$\det(\mathbf{J}) = \det\begin{pmatrix} \partial y'/\partial r' & \partial y'/\partial \psi \\ \partial z'/\partial r' & \partial z'/\partial \psi \end{pmatrix} = \det\begin{pmatrix} \cos\psi & -r'\sin\psi \\ \sin\psi & r'\cos\psi \end{pmatrix} = r'.$$

Hence, we obtain

$$\tilde{p}(r,\varphi,\theta,t) = \tilde{b}(\varphi,\theta)\,\frac{\hat{Q}}{r}\,e^{j(\omega t - kr)},$$

where

$$\begin{aligned}\tilde{b}(\varphi,\theta) &= \frac{1}{\hat{Q}}\int_0^a\int_{-\pi}^{\pi} q_s(r'\cos\psi, r'\sin\psi)\,e^{jkr'(\sin\varphi\cos\theta\cos\psi + \sin\theta\sin\psi)}\,r'd\psi dr' \\ &= \frac{1}{\hat{Q}}\int_0^a\int_{-\pi}^{\pi} q_s(r'\cos\psi, r'\sin\psi)\,e^{jr'(k_y\cos\psi + k_z\sin\psi)}\,r'd\psi dr'\end{aligned}$$

and

$$\hat{Q} = \int_0^a \int_{-\pi}^{\pi} \hat{q}_s(r'\cos\psi, r'\sin\psi)\, r' d\psi dr'$$

with

$$q_s(r'\cos\psi, r'\sin\psi) = \hat{q}_s(r'\cos\psi, r'\sin\psi)\, e^{j\alpha(r'\cos\psi, r'\sin\psi)}.$$

If the aperture function possesses circular symmetry, i.e.

$$q_s(r'\cos\psi, r'\sin\psi) = \overline{q}_s(r'),$$

the complex beam pattern can be written as

$$\overline{b}(\beta) = \frac{1}{\hat{Q}} \int_0^a \int_{-\pi}^{\pi} \overline{q}_s(r')\, e^{jkr'\sin\beta\sin\psi}\, r' d\psi dr',$$

where β denotes the angle between the vector **r** and the x-axis.

For $\overline{q}_s(r')=1$ we obtain

$$\overline{b}(\beta)=\frac{1}{\pi a^2}\int_0^a\left(\int_{-\pi}^{\pi} e^{jkr'\sin\beta\sin\psi}\,d\psi\right)r'dr'=\frac{2}{a^2}\int_0^a J_0(kr'\sin\beta)\,r'dr',$$

where

$$J_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(x\sin\gamma-n\gamma)}d\gamma=\sum_{l=0}^{\infty}\frac{(-1)^l}{l!(n+l)!}\left(\frac{x}{2}\right)^{n+2l}$$

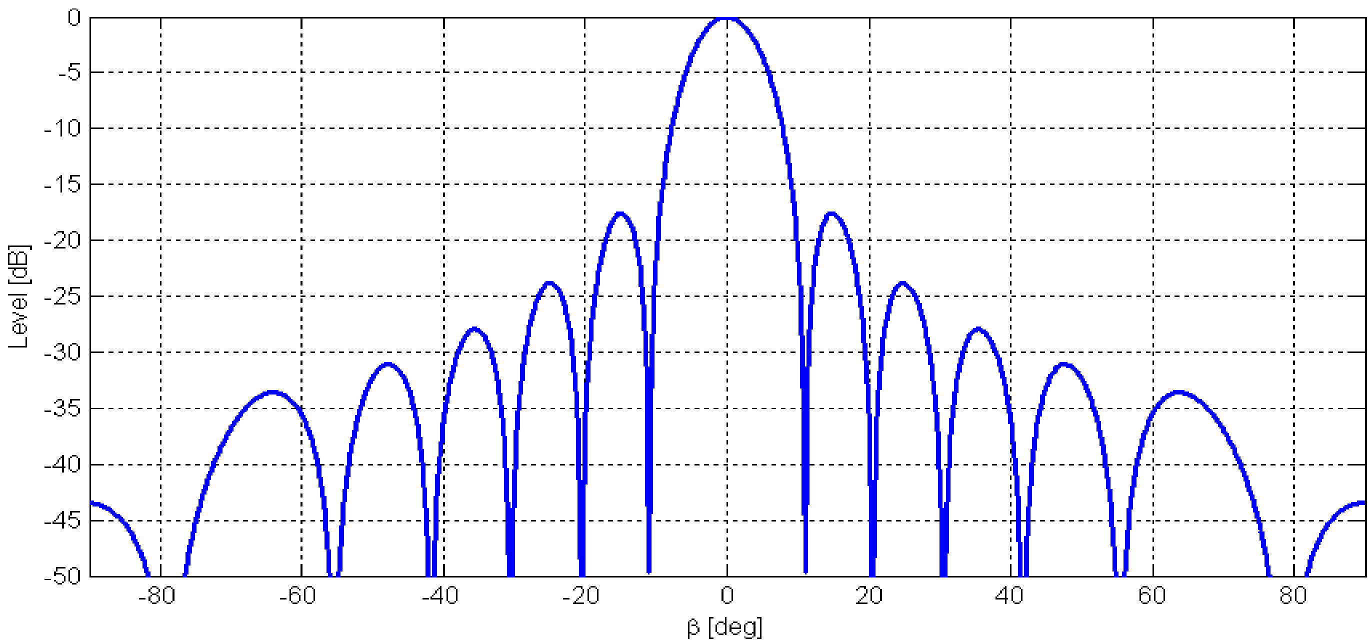
denotes the n-th order Bessel function of the first kind. Finally, exploiting the identity

$$\frac{d}{dx}\left(x^n J_n(x)\right)=x^n J_{n-1}(x)\;\Rightarrow\;\int x^n J_{n-1}(x)\,dx=x^n J_n(x)+C$$

the complex beam pattern can be represented by

$$\overline{b}(\beta)=\frac{2}{k^2a^2\sin^2\beta}\,x\,J_1(x)\Big|_0^{ka\sin\beta}=\frac{2}{ka\sin\beta}J_1(ka\sin\beta).$$

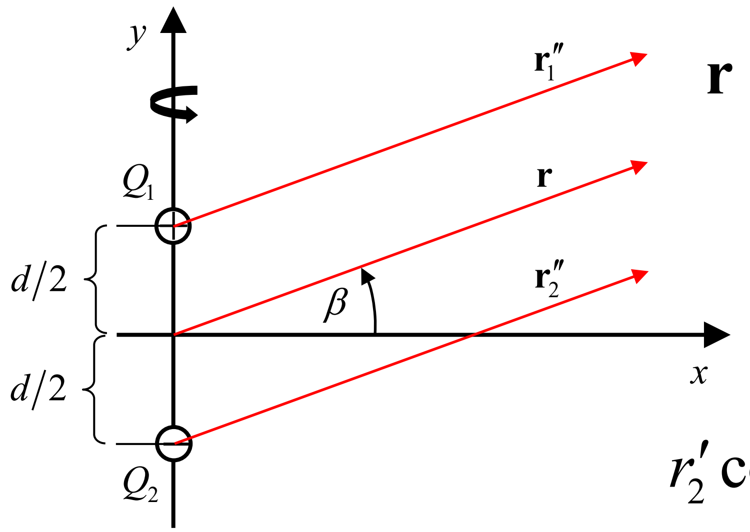
Typical beam pattern $\overline{B}(\beta)$ of a circular aperture with q_s= const.



3.1.3 Discrete Apertures

Dipole

A dipole consists of two in phase opposition working point sources of equal strength and displaced by d which is small compared to the wave length, i.e. $kd = 2\pi d/\lambda \ll 1$.



$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}, \ \mathbf{r}_1' = \begin{pmatrix} 0 \\ d/2 \end{pmatrix}, \ \mathbf{r}_2' = \begin{pmatrix} 0 \\ -d/2 \end{pmatrix}$$

$$r_1' \cos\left(\measuredangle(\mathbf{r}, \mathbf{r}_1')\right) = \frac{d}{2} \cdot \frac{y}{r} = \frac{d}{2} \sin\beta$$

$$r_2' \cos\left(\measuredangle(\mathbf{r}, \mathbf{r}_2')\right) = -\frac{d}{2} \cdot \frac{y}{r} = -\frac{d}{2} \sin\beta$$

Supposing $Q_1 = -Q_2 = \hat{q}$ the pressure can be determined by

$$p(\mathbf{r},t) = \frac{e^{j(\omega t - kr)}}{r}\hat{q}\left(e^{jkr_1'\cos(\measuredangle(\mathbf{r},\mathbf{r}_1'))} - e^{jkr_2'\cos(\measuredangle(\mathbf{r},\mathbf{r}_2'))}\right)$$

or in polar coordinates by

$$\begin{aligned}\tilde{p}(r,\beta,t) &= \frac{e^{j(\omega t - kr)}}{r}\hat{q}\left(e^{jkd/2\sin\beta} - e^{-jkd/2\sin\beta}\right)\\ &= 2j\sin\left(\frac{kd}{2}\sin\beta\right)\hat{q}\frac{e^{j(\omega t - kr)}}{r}.\end{aligned}$$

Since $kd \ll 1$ we can approximately write

$$\sin\left(\frac{kd}{2}\sin\beta\right) \approx \frac{kd}{2}\sin\beta.$$

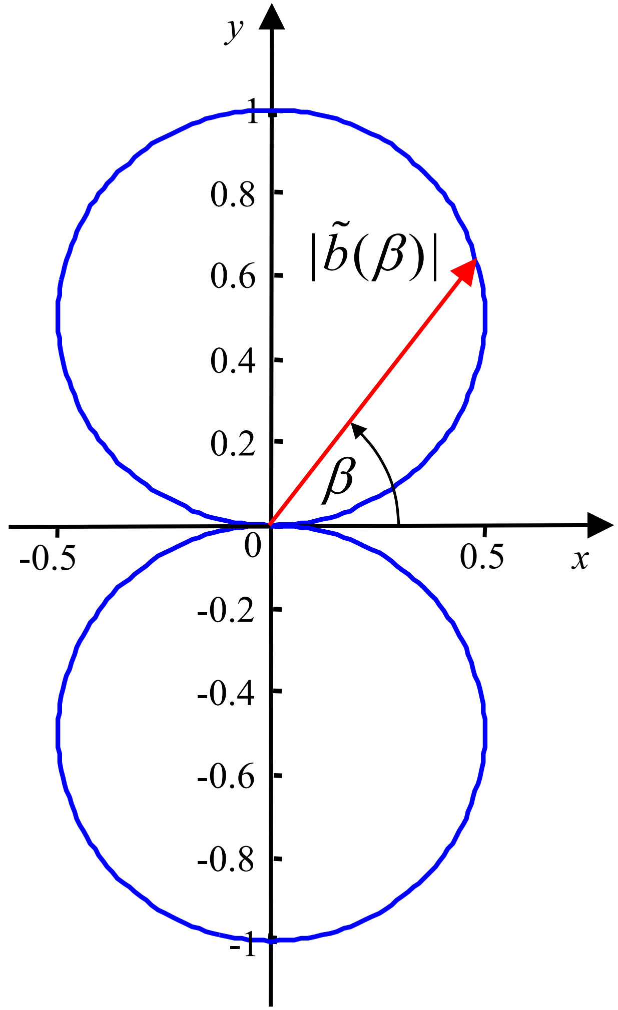
Hence, after introducing the beam pattern

$$\tilde{b}(\beta) = \sin\beta$$

the pressure can approximately be calculated by

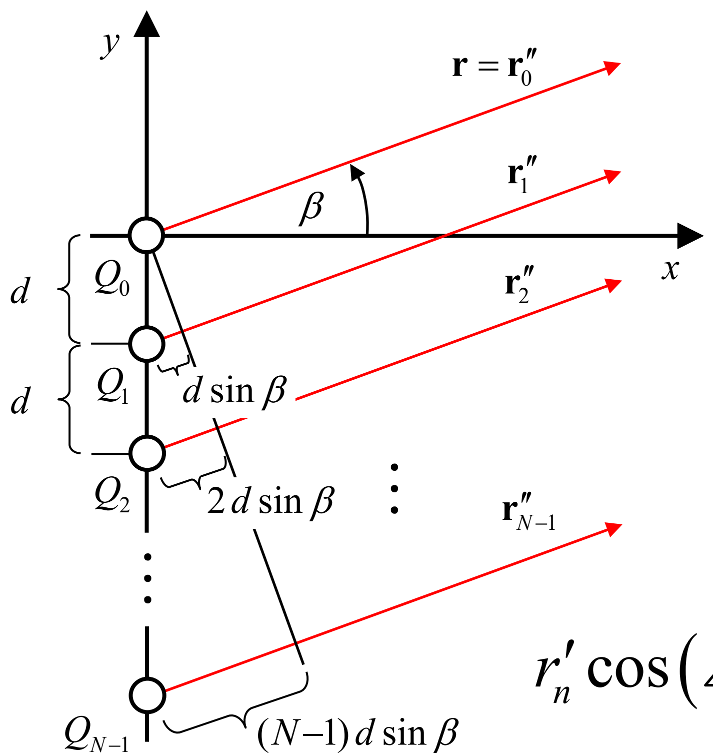
$$\tilde{p}(r,\beta,t) \approx jkd\,\hat{q}\,\tilde{b}(\beta)\,\frac{e^{j(\omega t - kr)}}{r}.$$

The beam pattern of a dipole which possesses an "eight characteristic" is shown in the diagram on the right side.



Linear Array

The aperture function for a discrete linear array can be interpreted as a sampled version of a corresponding continuous linear aperture function, i.e.



$$q_l(y) = \sum_{n=0}^{N-1} Q_n\,\delta(y - y_n').$$

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix},\quad \mathbf{r}_1' = \begin{pmatrix} 0 \\ y_n' \end{pmatrix},\quad \mathbf{r}_2' = \begin{pmatrix} 0 \\ -nd \end{pmatrix}$$

$$r_n' \cos\left(\measuredangle(\mathbf{r},\mathbf{r}_n')\right) = \frac{\mathbf{r}^T\mathbf{r}_n'}{r} = \frac{y\,y_n'}{r} = -nd\sin\beta$$

Hence, the pressure wave field can be described by

$$p(\mathbf{r},t) = \frac{e^{j(\omega t - kr)}}{r} \int_{-\infty}^{\infty} q_l(y')\, e^{jk\frac{y}{r}y'}\, dy'$$

$$= \frac{e^{j(\omega t - kr)}}{r} \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} Q_n\, \delta(y' - y_n')\, e^{jk\frac{y}{r}y'}\, dy'$$

$$= \frac{e^{j(\omega t - kr)}}{r} \sum_{n=0}^{N-1} Q_n\, e^{jk\frac{y}{r}y_n'}$$

or in polar coordinates by

$$\tilde{p}(r,\beta,t) = \frac{e^{j(\omega t - kr)}}{r} \sum_{n=0}^{N-1} Q_n e^{jk\, y_n' \sin\beta}\,.$$

Moreover, after introducing the complex beam pattern

$$\tilde{b}(\beta) = \frac{1}{\hat{Q}} \sum_{n=0}^{N-1} Q_n e^{jky'_n \sin\beta} \quad \text{with} \quad Q_n = \hat{Q}_n e^{j\alpha_n} \quad \text{and} \quad \hat{Q} = \sum_{n=0}^{N-1} \hat{Q}_n$$

the wave field can be written as

$$\tilde{p}(r,\beta,t) = \tilde{b}(\beta)\,\frac{\hat{Q}}{r}\,e^{j(\omega t - kr)}.$$

For $Q_0 = Q_1 = \ldots = Q_{N-1} = 1$, i.e. $\hat{Q}_n = 1$, $\alpha_n = 0$, and $y'_n = -nd$ the complex beam pattern simplifies to

$$\tilde{b}(\beta) = \frac{1}{N} \sum_{n=0}^{N-1} e^{-jknd\sin\beta}.$$

Furthermore, exploiting the well know result

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q} \quad \text{(finite geometric series)}$$

the complex beam pattern can be written in closed form as

$$\tilde{b}(\beta) = \frac{1}{N}\frac{1-e^{-jkNd\sin\beta}}{1-e^{-jkd\sin\beta}}.$$

By taking the magnitude of $\tilde{b}(\beta)$ we obtain

$$|\tilde{b}(\beta)| = \frac{1}{N}\left|\frac{e^{jkNd/2\sin\beta}-e^{-jkNd/2\sin\beta}}{e^{jkd/2\sin\beta}-e^{-jkd/2\sin\beta}}\cdot\frac{e^{-jkNd/2\sin\beta}}{e^{-jkd/2\sin\beta}}\right|$$

$$= \frac{1}{N}\left|\frac{\sin(Nkd/2\sin\beta)}{\sin(kd/2\sin\beta)}\right|$$

Assignment 7:

Develop a Matlab program that determines

$$\tilde{B}(\beta) = 20\log_{10}|\tilde{b}(\beta)| = 20\log_{10}\left|\frac{1}{\hat{Q}}\sum_{n=0}^{N-1} Q_n e^{-jknd\sin\beta}\right|$$

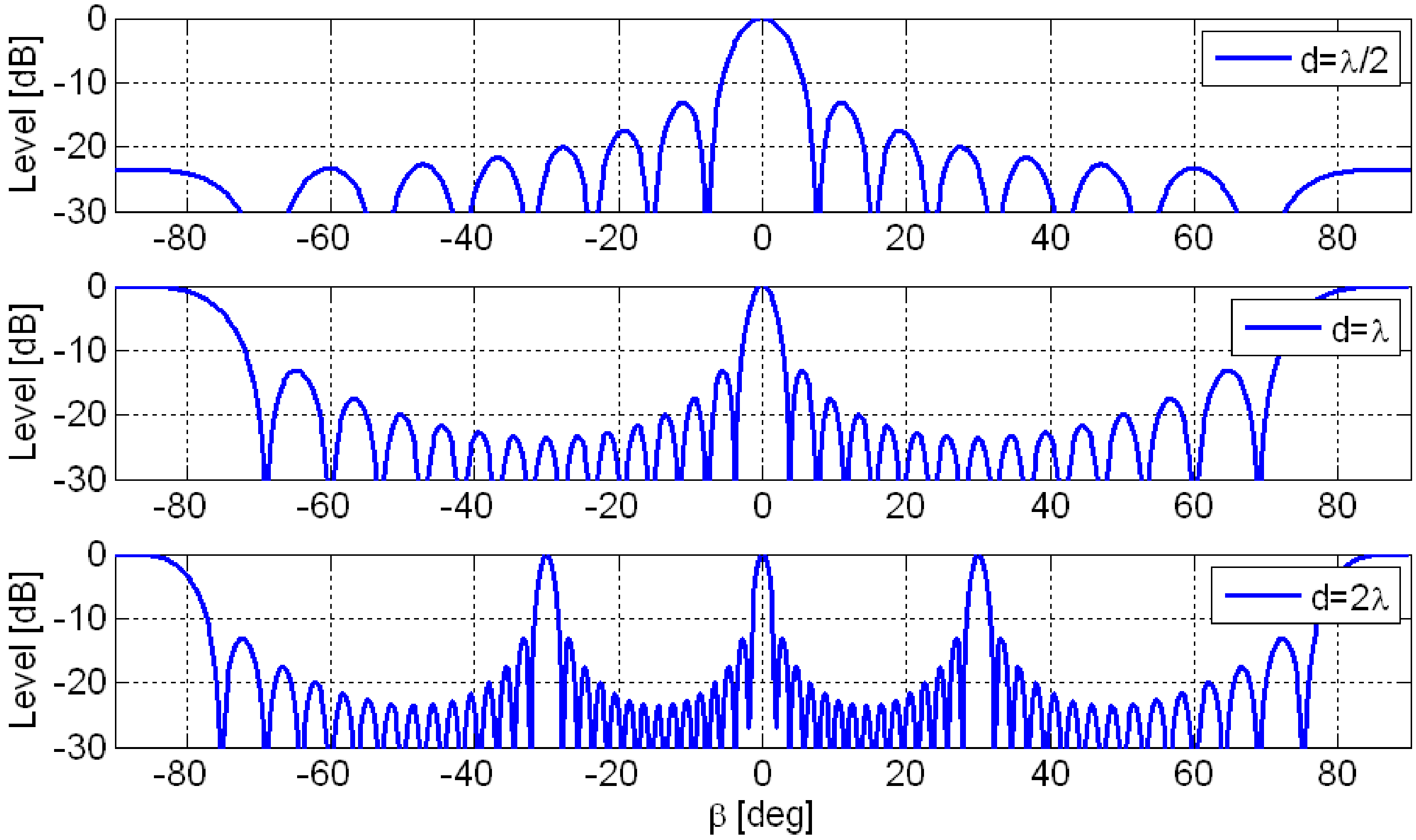
with

$$Q_n = \hat{Q}_n e^{j\alpha_n} \quad \text{and} \quad \hat{Q} = \sum_{n=0}^{N-1} \hat{Q}_n$$

for the following parameters.

1) $d = \lambda/2, \lambda, 2\lambda$ element spacing
2) Amplitude shading, e.g. chebwin
3) Linear phase shading, i.e. electronic steering
4) Parabolic phase shading, i.e. beam shaping

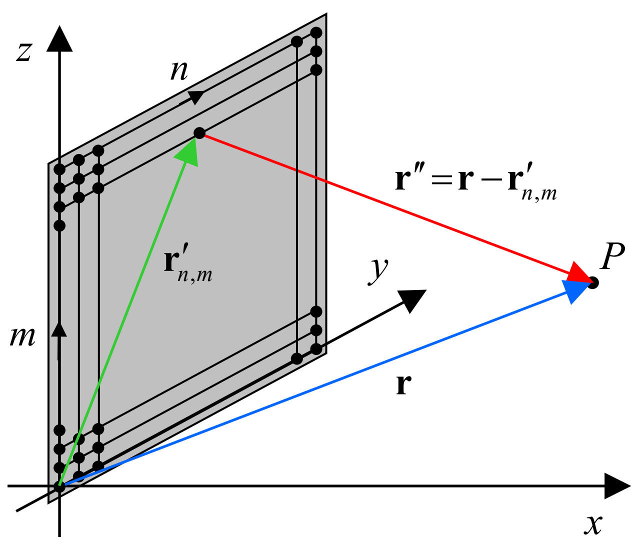
Typical beam pattern $\tilde{B}(\beta)$ of a linear array with $Q_n = \text{const.}$



Planar Array

The aperture function for a discrete planar array can be also thought of as a sampled version of a continuous planar aperture function, i.e.

$$q_s(y,z) = \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} Q_{n,m}\,\delta(y-y'_n, z-z'_m).$$



$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{r}'_{n,m} = \begin{pmatrix} 0 \\ y'_n \\ z'_m \end{pmatrix} = \begin{pmatrix} 0 \\ nd_y \\ md_z \end{pmatrix}$$

$$r'_{n,m}\cos\left(\measuredangle(\mathbf{r},\mathbf{r}'_{n,m})\right) = \frac{y\,y'_n + z\,z'_m}{r}$$

Thus, the pressure wave field can be written as

$$p(\mathbf{r},t) = \frac{e^{j(\omega t - kr)}}{r} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_s(y',z')\, e^{jk\left(\frac{y}{r}y' + \frac{z}{r}z'\right)} dy'dz'$$

$$= \frac{e^{j(\omega t - kr)}}{r} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} Q_{n,m}\,\delta(y'-y'_n, z'-z'_m)\, e^{jk\left(\frac{y}{r}y' + \frac{z}{r}z'\right)} dy'dz'$$

$$= \frac{e^{j(\omega t - kr)}}{r} \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} Q_{n,m}\, e^{jk\left(\frac{y}{r}y'_n + \frac{z}{r}z'_m\right)}$$

which by employing

$$y/r = \sin\varphi\cos\theta \quad \text{and} \quad z/r = \sin\theta$$

can be reformulated in spherical coordinates to

$$\tilde{p}(r,\varphi,\theta,t) = \frac{e^{j(\omega t - kr)}}{r} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} Q_{n,m} \; e^{jk(\sin\varphi\cos\theta\, y'_n + \sin\theta\, z'_m)}$$

$$= \tilde{b}(\varphi,\theta)\, \frac{\hat{Q}}{r}\, e^{j(\omega t - kr)},$$

where $\tilde{b}(\varphi,\theta)$ denotes the complex beam pattern defined by

$$\tilde{b}(\varphi,\theta) = \frac{1}{\hat{Q}} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} Q_{n,m} \; e^{jk(\sin\varphi\cos\theta\, y'_n + \sin\theta\, z'_m)}$$

with

$$Q_{n,m} = \hat{Q}_{n,m}\, e^{j\alpha_{n,m}} \quad \text{and} \quad \hat{Q} = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \hat{Q}_{n,m}.$$

For $Q_{n,m}=1$ and $y'_n = nd_y$, $z'_n = md_z$ the beam pattern simplifies to

$$\tilde{b}(\varphi,\theta) = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=0}^{M-1} e^{jk(nd_y \sin\varphi\cos\theta + md_z \sin\theta)}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{jknd_y \sin\varphi\cos\theta} \frac{1}{M}\sum_{m=0}^{M-1} e^{jkmd_z \sin\theta}.$$

Exploiting again the closed form representation for finite geo-metric series we can state

$$\tilde{b}(\varphi,\theta) = \frac{1}{N}\frac{1-e^{jkNd_y \sin\varphi\cos\theta}}{1-e^{jkd_y \sin\varphi\cos\theta}} \cdot \frac{1}{M}\frac{1-e^{jkMd_z \sin\theta}}{1-e^{jkd_z \sin\theta_z}}.$$

Finally, after some reformulations

$$\tilde{b}(\varphi,\theta)=\frac{1}{N}\frac{e^{-jkN/2\sin\varphi\cos\theta\, d_y}-e^{jkN/2\sin\varphi\cos\theta\, d_y}}{e^{-jk/2\sin\varphi\cos\theta\, d_y}-e^{jk/2\sin\varphi\cos\theta\, d_y}}\frac{e^{jkN/2\sin\varphi\cos\theta\, d_y}}{e^{jk/2\sin\varphi\cos\theta\, d_y}}\times$$

$$\times\frac{1}{M}\frac{e^{-jkM/2\sin\theta\, d_z}-e^{jkM/2\sin\theta\, d_z}}{e^{-jk/2\sin\theta\, d_z}-e^{jk/2\sin\theta\, d_z}}\frac{e^{jkM/2\sin\theta\, d_z}}{e^{jk/2\sin\theta\, d_z}}$$

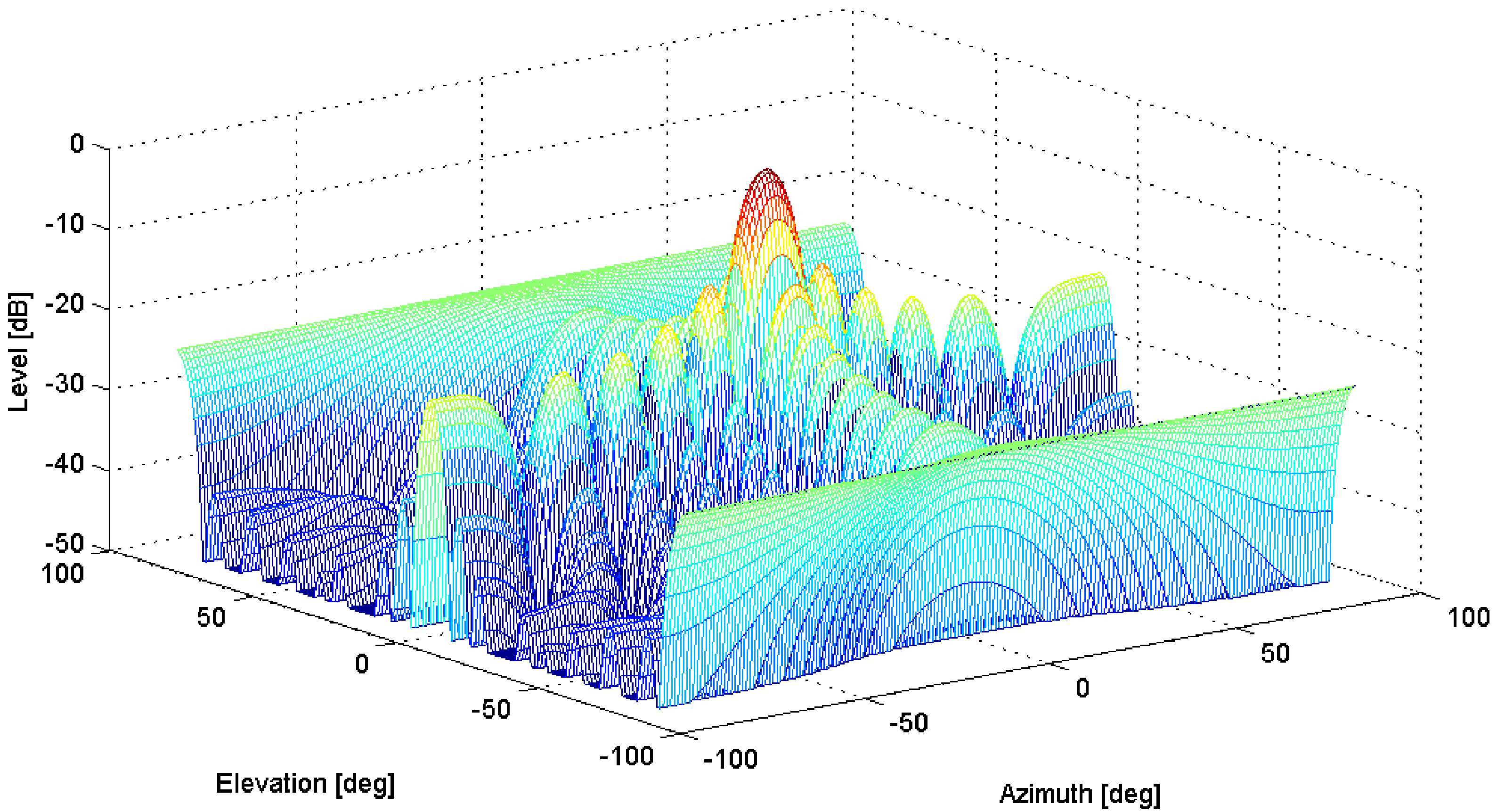
the magnitude of $\tilde{b}(\varphi,\theta)$ can be expressed by

$$|\tilde{b}(\varphi,\theta)|=\frac{1}{NM}\left|\frac{\sin(kN/2\sin\varphi\cos\theta\, d_y)}{\sin(k/2\sin\varphi\cos\theta\, d_y)}\cdot\frac{\sin(kM/2\sin\theta\, d_z)}{\sin(k/2\sin\theta\, d_z)}\right|$$

and in decibels by

$$\tilde{B}(\varphi,\theta)=20\log_{10}|\tilde{b}(\varphi,\theta)|\,.$$

Typical beam pattern $\tilde{B}(\varphi,\theta)$ of a rectangular planar array with $Q_{n,m} = \text{const.}$



Analogy of transmitter/receiver characteristics

The laws of the sound radiation treated before can be transferred to the sound reception.

That is, a sound wave emitted at a point *P* and measured via an arrangement of sound-sensitive sensors, e.g. an array of hydrophones, provides a superimposed signal that possesses the same location (direction and range) dependence on *P* as the sound pressure in the case of sound radiation.

In particular the same principles for the near-field and far-field transition as well as for the beam pattern apply, cf. to this also Chapter 5.

3.1.4 Array Gain and Directivity Index

When projectors resp. hydrophones are assembled in arrays the transmission power can be focused resp. the signal-to-noise ratio can be improved.

Transmitting Array Gain

Let $S(\varphi,\theta)$ denote the angular characteristic of the sound field emitted by each individual projector of an array and let $\tilde{b}(\varphi,\theta)$ represent the beam pattern of the array if omnidirectional projectors would be used, the array gain (AG) is defined as

$$AG = 10\log_{10}\left(\frac{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} S(\varphi,\theta)\cos\theta\, d\theta d\varphi}{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} S(\varphi,\theta)\left|\tilde{b}(\varphi,\theta)\right|^2 \cos\theta\, d\theta d\varphi}\right).$$

Receiving Array Gain

Now, $N(\varphi,\theta)$ represents the angular distribution of the noise field and $\tilde{b}(\varphi,\theta)$ the spatial filter characteristic of an array of omnidirectional hydrophones. Hence, the array gain (AG) is defined as

$$AG = 10\log_{10}\left(\frac{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} N(\varphi,\theta)\cos\theta\, d\theta d\varphi}{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2} N(\varphi,\theta)\left|\tilde{b}(\varphi,\theta)\right|^2 \cos\theta\, d\theta d\varphi}\right).$$

Directivity Index (receive and transmit)

If $S(\varphi,\theta)$ is omnidirectional resp. $N(\varphi,\theta)$ is isotropic the array gain simplifies to the so-called directivity index

$$DI = 10\log_{10}\left(4\pi \Big/ \int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\left|\tilde{b}(\varphi,\theta)\right|^2 \cos\theta\, d\theta d\varphi\right)$$

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