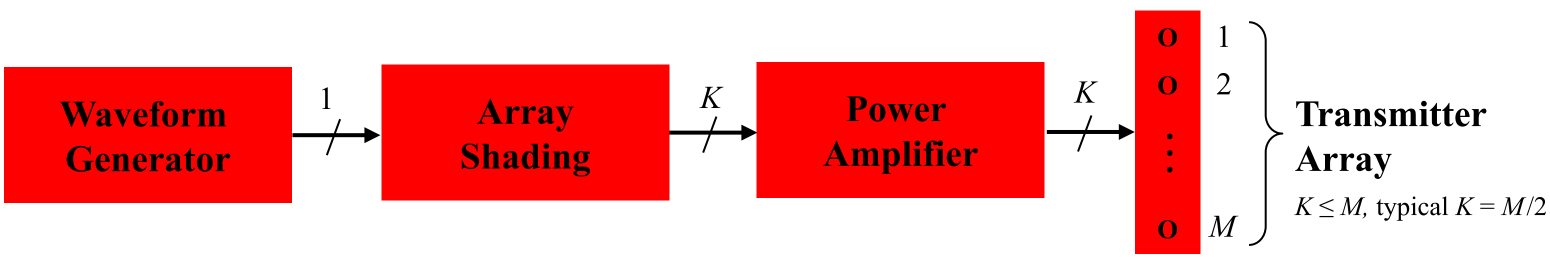
Underwater Acoustics and Sonar Signal Processing

Contents

4 Sonar Signal Processing

4.1 Introduction

The following diagram summarizes the components usually employed in Sonar systems for sound transmission.



Waveform Generator

- CW (continuous waveforms), e.g. sinusoidal pulse with rectangular or Gaussian envelope
- FM (frequency modulated) waveforms, e.g.

linear FM (LFM), hyperbolic FM (HFM) and Doppler sensitive FM (DFM)

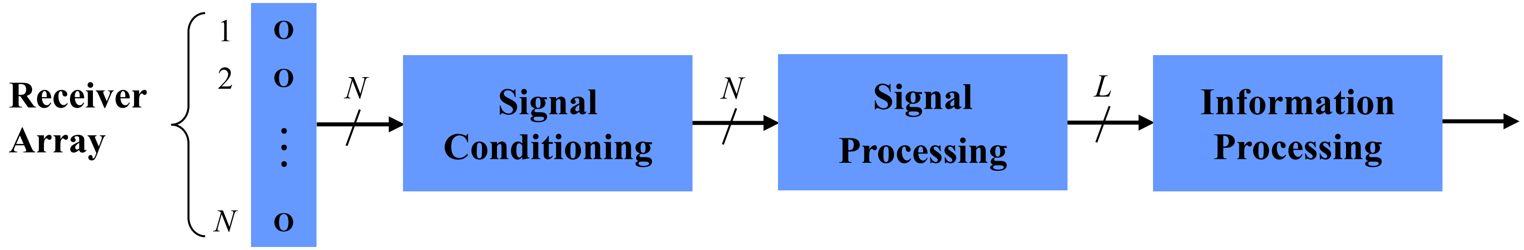
Array Shading

- Amplitude shading for side-lobe suppression
- Complex shading (amplitude shading and phase shifting / time delays) for main-lobe steering, shaping and broadening

Power Amplifier / Impedance Matching

- Switching amplifiers to achieve high source levels
- Linear amplifiers if moderate source levels are sufficient but an enhanced coherence of consecutive pulses is required, e.g. as in synthetic aperture sonar SAS applications
- Impedance matching networks that supplies an optimal coupling of the amplifiers to the transducers

The receiver electronic measuring and information processing chain consists of the following components.



Signal Conditioning

- Preamplifier and Band-Pass Filter
- Automatic Gain Control (AGC) and/or (Adaptive) Time variable Gain ((A)TVG)
- Quadrature Demodulation (analog or digitally)
- Anti-aliasing Filter and Analog-to-Digital Conversion with 16 up to 24 bits.

Signal Processing

- Matched Filtering / Pulse Compression
- Conventional motion compensated near and far field beam-forming in time or frequency domain
- Synthetic aperture beamforming including micro-navigation and auto-focusing
- High resolution source localization techniques (MVDR, MUSIC, ESPRIT, etc.)

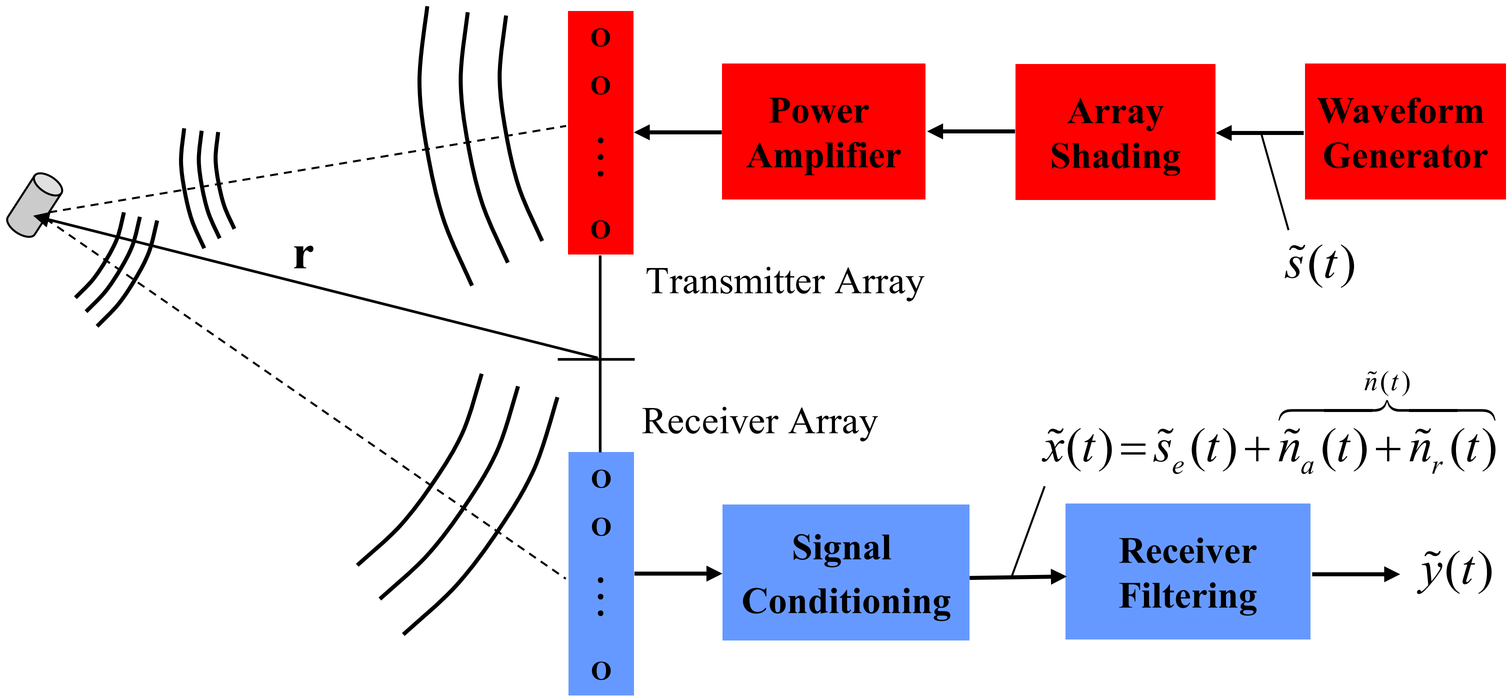
Information Processing

- Image Formation (range and azimuth decimation/interpolation)
- Image Fusion (multi-ping and/or multi-aspect mode)
- Computer aided Target detection/classification (semi-automatic Operator support) and Autonomous Target Recognition (ATR)

4.2 Matched Filtering

4.2.1 Matched Filtering for Band-Pass Signals

A typical Transmission/Reception Scenario is depicted below,



where

$\tilde{s}(t)$ denotes the transmitted band-pass signal with $\tilde{s}(t)=0$ for $t \notin [0,T]$ and $\|\tilde{s}\|^2 = \int \tilde{s}^2(t)\,dt < \infty$, i.e. finite signal energy,

$\tilde{s}_e(t)$ denotes the received echo signal with $\tilde{s}_e(t) = a\,\tilde{s}(t-\tau)$, where $a \neq 0$ models the propagation and reflection losses and $\tau = 2|\mathbf{r}|/c$ represents the travel time

$\tilde{n}(t)$ denotes a wide sense stationary noise process with $\tilde{n}(t) = \tilde{n}_a(t) + \tilde{n}_r(t)$, where $\tilde{n}_a(t)$ and $\tilde{n}_r(t)$ describe the ambient noise and receiver noise respectively.

Now, we want to determine the impulse response $\tilde{h}(t)$ of a stable receiver filter, i.e.

$$\int \left|\tilde{h}(t)\right| dt < \infty,$$

such that the output signal

$$\tilde{y}(t) = \int \tilde{h}(t')\tilde{x}(t-t')\,dt' = \int \tilde{h}(t')\tilde{s}_e(t-t')\,dt' + \int \tilde{h}(t')\tilde{n}(t-t')\,dt'$$

possesses a maximum signal-to-noise ratio at $t = \tau$. Thus

$$\tilde{\gamma}(\tilde{h}) = \frac{\left(\int \tilde{h}(t')\tilde{s}_e(\tau - t')\,dt'\right)^2}{\mathrm{E}\left(\int \tilde{h}(t')\tilde{n}(\tau - t')\,dt'\right)^2} = \frac{\tilde{s}^2_{e,\tilde{h}}(\tau)}{\mathrm{E}\left(\tilde{n}^2_{\tilde{h}}(\tau)\right)} = \frac{\tilde{s}^2_{e,\tilde{h}}(\tau)}{\sigma^2_{\tilde{n}_{\tilde{h}}}}$$

has to be maximized, where

$$\tilde{s}_{e,\tilde{h}}(\tau) = \int \tilde{h}(t')\,\tilde{s}_e(\tau - t')\,dt' = a\int \tilde{h}(t')\,\tilde{s}(-t')\,dt'$$

and

$$\tilde{n}_{\tilde{h}}(\tau) = \int \tilde{h}(t')\,\tilde{n}(\tau - t')\,dt'.$$

The second order moment (correlation) function of the zero mean wide sense stationary process $\tilde{n}_{\tilde{h}}(\tau)$ can be written as

$$\begin{aligned} r_{\tilde{n}_{\tilde{h}}\tilde{n}_{\tilde{h}}}(\tau) &= \mathrm{E}\left(\tilde{n}_{\tilde{h}}(t+\tau)\tilde{n}_{\tilde{h}}(t)\right) \\ &= \mathrm{E}\left(\int \tilde{h}(t')\tilde{n}(t+\tau-t')\,dt' \cdot \int \tilde{h}(t'')\tilde{n}(t-t'')\,dt''\right) \\ &= \iint \tilde{h}(t')\tilde{h}(t'')\,\mathrm{E}\left(\tilde{n}(t+\tau-t')\tilde{n}(t-t'')\right)dt'dt'' \\ &= \iint \tilde{h}(t')\tilde{h}(t'')\,r_{\tilde{n}\tilde{n}}(\tau - t' + t'')\,dt'dt''. \end{aligned}$$

Applying the Wiener-Khintchine Theorem the power spectral density function of $\tilde{n}_{\tilde{h}}(\tau)$ is given by

$$
\begin{aligned}
R_{\tilde{n}_{\tilde{h}}\tilde{n}_{\tilde{h}}}(\omega) &= \int r_{\tilde{n}_{\tilde{h}}\tilde{n}_{\tilde{h}}}(\tau) e^{-j\omega\tau} d\tau \\
&= \iiint \tilde{h}(t')\tilde{h}(t'')\, r_{\tilde{n}\tilde{n}}(\tau - t' + t'')\, e^{-j\omega\tau} dt'dt''d\tau \\
&= \iint \tilde{h}(t')\tilde{h}(t'')\left(\int r_{\tilde{n}\tilde{n}}(\tau - t' + t'')\, e^{-j\omega\tau} d\tau\right) dt'dt'' \\
&= \iint \tilde{h}(t')\tilde{h}(t'')\, R_{\tilde{n}\tilde{n}}(\omega) e^{-j\omega(t'-t'')} dt'dt'' \\
&= R_{\tilde{n}\tilde{n}}(\omega) \int \tilde{h}(t')\, e^{-j\omega t'} dt' \cdot \int \tilde{h}(t'')\, e^{j\omega t''} dt'' \\
&= R_{\tilde{n}\tilde{n}}(\omega)\tilde{H}(\omega)\tilde{H}^*(\omega) = \left|\tilde{H}(\omega)\right|^2 R_{\tilde{n}\tilde{n}}(\omega).
\end{aligned}
$$

The variance (power) of $\tilde{n}_{\tilde{h}}(\tau)$ can now be described in terms of the power spectral density function $R_{\tilde{n}\tilde{n}}(\omega)$ as follows.

$$\sigma^2_{\tilde{n}_{\tilde{h}}} = \mathrm{E}\left(\tilde{n}^2_{\tilde{h}}(t)\right) = r_{\tilde{n}_{\tilde{h}}\tilde{n}_{\tilde{h}}}(0) = \frac{1}{2\pi}\int R_{\tilde{n}_{\tilde{h}}\tilde{n}_{\tilde{h}}}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int \left|\tilde{H}(\omega)\right|^2 R_{\tilde{n}\tilde{n}}(\omega)\,d\omega$$

Since $\tilde{n}(t)$ is supposed to exhibit a constant power spectral density level

$$R_{\tilde{n}\tilde{n}}(\omega) = N_0/2$$

within the frequency band of interest, we finally obtain

$$\sigma^2_{\tilde{n}_{\tilde{h}}} = \mathrm{E}\left(\tilde{n}^2_{\tilde{h}}(t)\right) = N_0/2 \cdot \frac{1}{2\pi}\int \left|\tilde{H}(\omega)\right|^2 d\omega\,.$$

Exploiting Parseval's Formula, i.e.

$$\left\|\tilde{h}\right\|^2 = \int \left|\tilde{h}(t)\right|^2 dt = \frac{1}{2\pi}\int \left|\tilde{H}(\omega)\right|^2 d\omega = \frac{1}{2\pi}\left\|\tilde{H}(\omega)\right\|^2,$$

we have to maximize

$$\tilde{\gamma}(\tilde{h}) = \frac{\left(a\int \tilde{h}(t)\tilde{s}(-t)\,dt\right)^2}{N_0/2\cdot\int\left|\tilde{h}(t)\right|^2 dt} = a^2\frac{\left\|\tilde{s}\right\|^2}{N_0/2}\frac{\left(\int \tilde{h}(t)\tilde{s}(-t)\,dt\right)^2}{\left\|\tilde{h}\right\|^2\left\|\tilde{s}\right\|^2}$$

$$= a^2\frac{\left\|\tilde{s}\right\|^2}{N_0/2}\frac{\left(\int \tilde{h}(t)\hat{s}(t)\,dt\right)^2}{\left\|\tilde{h}\right\|^2\left\|\hat{s}\right\|^2}$$

with $\hat{s}(t) = \tilde{s}(-t)$ and $\left\|\tilde{s}\right\|^2 = \left\|\hat{s}\right\|^2$.

Application of the Cauchy-Schwarz inequality

$$\left| \int f_1(t) f_2^*(t) dt \right|^2 \leq \int \left| f_1(t) \right|^2 dt \cdot \int \left| f_2(t) \right|^2 dt$$

tells us, that the maximum is achieved for

$$\tilde{h}_{opt}(t) = c\,\hat{s}(t) = c\,\tilde{s}(-t).$$

Thus, the impulse response of the optimum receiver filter is matched to the transmitter signal. The optimum receiver filter is therefore called matched filter.

Finally, the maximum signal-to-noise ratio is given by

$$\tilde{\gamma}(\tilde{h}_{opt}) = a^2 \frac{\left\| \tilde{s} \right\|^2}{N_0/2}.$$

For $c = 1$ the matched filter output can be expressed by

$$\tilde{y}(t) = \int \tilde{h}_{opt}(t')\,\tilde{x}(t-t')\,dt' = \int \tilde{s}(-t')\,\tilde{x}(t-t')\,dt'$$
$$= \int \tilde{x}(t+t'')\,\tilde{s}(t'')\,dt'' = r_{\tilde{x}\tilde{s}}(t),$$

where $r_{\tilde{x}\tilde{s}}(t)$ denotes the cross-correlation function.

Hence, the matched filtering process can be interpreted as the correlation of the input signal $\tilde{x}(t)$ with the expected (transmitted) signal $\tilde{s}(t)$.

The point target response of the receiver is defined by

$$p(t) = \tilde{h}(t) * \tilde{s}(t) = \int \tilde{h}(t')\tilde{s}(t-t')\,dt'.$$

Applying the matched filter, we obtain

$$p_{opt}(t) = \tilde{h}_{opt}(t) * \tilde{s}(t) = \tilde{s}(-t) * \tilde{s}(t)$$
$$= \int \tilde{s}(-t')\tilde{s}(t-t')\,dt' = \int \tilde{s}(t+t'')\tilde{s}(t'')\,dt'' = r_{\tilde{s}\tilde{s}}(t).$$

Consequently, the point target response is determined by the autocorrelation function of the transmitter signal if matched filtering is employed.

<u>*Example:*</u> *(CW-Pulse)*

Signal waveform:

$$\tilde{s}(t) = \mathrm{rect}_{T/2}(t)\cos(\omega_c t) \quad \text{with} \quad \mathrm{rect}_{T/2}(t) = \begin{cases} 1 & |t| < T/2 \\ 1/2 & |t| = T/2 \\ 0 & |t| > T/2 \end{cases}$$

Signal energy:

$$\|\tilde{s}\|^2 = \int \tilde{s}^2(t)\,dt = \int_{-T/2}^{T/2} \frac{1+\cos(2\omega_c t)}{2}\,dt \cong T/2$$

Matched Filter:

$$\tilde{h}_{opt}(t) = \tilde{s}(-t) = \tilde{s}(t)$$

Hence, the point target response can be expressed by

$$\begin{aligned} p_{opt}(t) &= \tilde{h}_{opt}(t) * \tilde{s}(t) = \tilde{s}(-t) * \tilde{s}(t) = r_{\tilde{s}\tilde{s}}(t) \\ &= \int \mathrm{rect}_{T/2}(t')\,\mathrm{rect}_{T/2}(t'+t)\cos(\omega_c t')\cos\big(\omega_c(t'+t)\big)\,dt' \\ &= \int \mathrm{rect}_{T/2}(t''-t/2)\,\mathrm{rect}_{T/2}(t''+t/2)\cdot \\ &\qquad \cos\big(\omega_c(t''-t/2)\big)\cos\big(\omega_c(t''+t/2)\big)\,dt'', \end{aligned}$$

where the substitution $t' = t'' - t/2$ with $dt' = dt''$ has been used.

After exploiting the identity

$$2\cos x \cos y = \left(\cos(x+y)+\cos(x-y)\right)$$

the point target response becomes

$$|t| \le T: \quad p_{opt}(t) = \int_{-d(t)}^{d(t)} \left(\cos(2\omega_c t'') + \cos(\omega_c t)\right) dt''/2$$

$$= \sin(2\omega_c t'')/(4\omega_c)\Big|_{-d(t)}^{d(t)} + t''\cos(\omega_c t)/2\Big|_{-d(t)}^{d(t)}$$

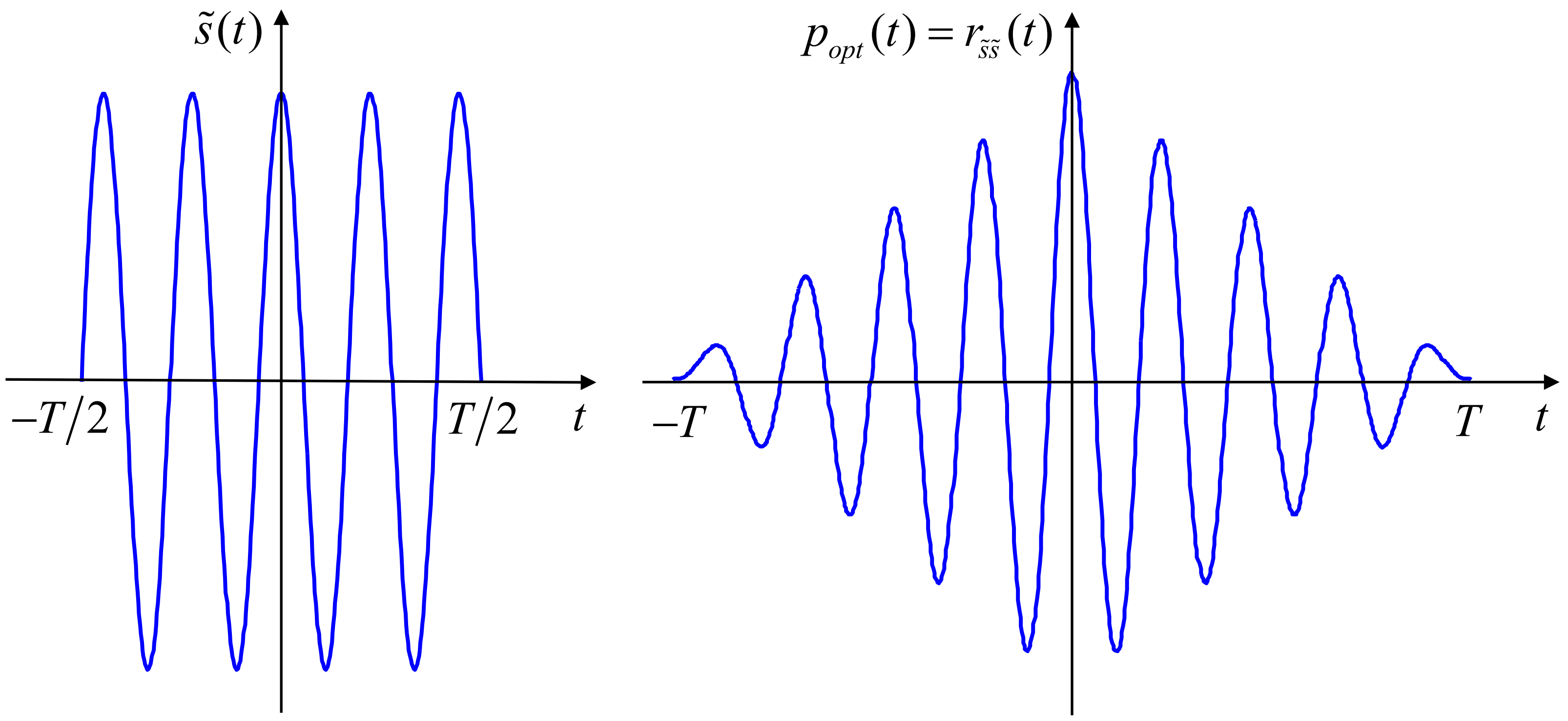
$$= \sin\left(2\omega_c d(t)\right)/(2\omega_c) + d(t)\cos(\omega_c t)$$

$$|t| > T: \quad p_{opt}(t) = 0,$$

where $d(t)$ denotes the triangular function

$$d(t) = \left(T - |t|\right)/2.$$

The first term can be neglected for $\omega_c \gg 1/T$ such that the second term with its triangular envelope remains.

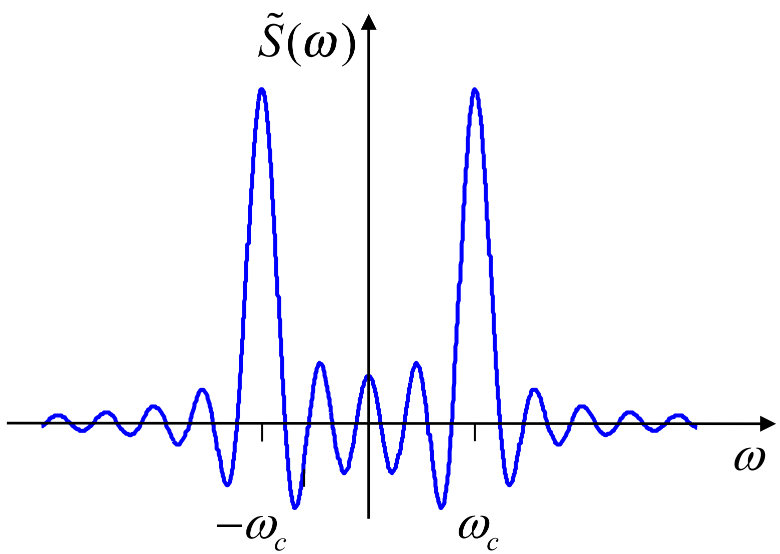


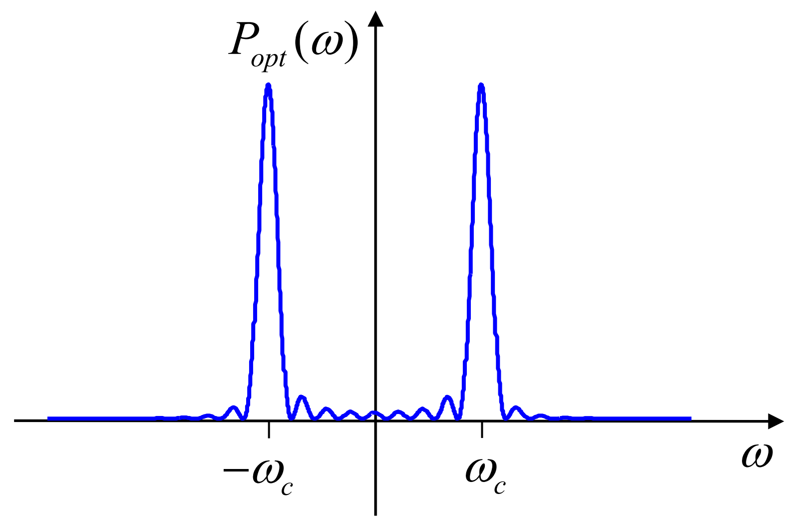
The Fourier transform of $\tilde{s}(t)$ and $p_{opt}(t)$ are given by

$$\tilde{S}(\omega) = \frac{T}{2}\left\{ \mathrm{si}\left(\frac{T}{2}(\omega - \omega_c) \right) + \mathrm{si}\left(\frac{T}{2}(\omega + \omega_c) \right) \right\}$$

and

$$P_{opt}(\omega) = R_{\tilde{S}\tilde{S}}(\omega) = \left| \tilde{S}(\omega) \right|^2 \quad \text{with} \quad \mathrm{si}(x) = \sin(x)/x\,.$$





The power spectral density function of the matched filtered white noise can be expressed by

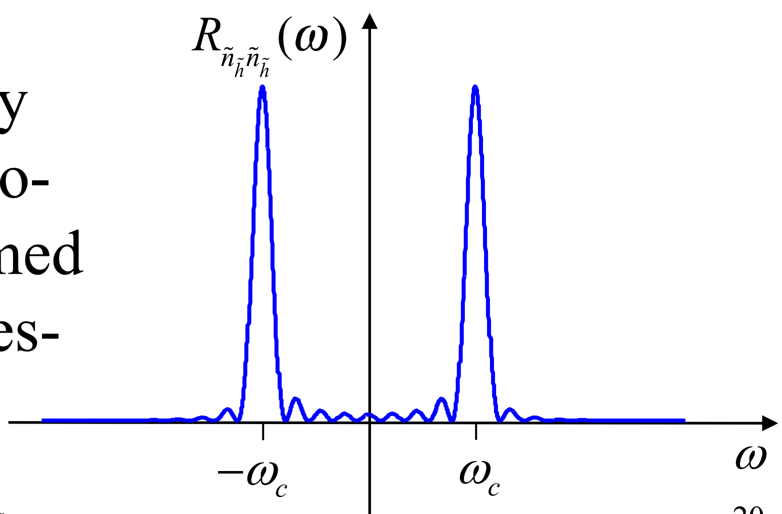
$$R_{\tilde{n}_{\tilde{h}}\tilde{n}_{\tilde{h}}}(\omega) = \left|\tilde{H}_{opt}(\omega)\right|^2 R_{\tilde{n}\tilde{n}}(\omega) = N_0/2\left|\tilde{S}(\omega)\right|^2 \propto P_{opt}(\omega),$$

where

$$\tilde{H}_{opt}(\omega) = \tilde{S}^*(\omega) \quad \text{and} \quad R_{\tilde{n}\tilde{n}}(\omega) = N_0/2$$

have been exploited.

Hence, the power spectral density function of the output noise is proportional to the Fourier transformed point target response, i.e. it possesses the same shape.



4.2.2 Quadrature Demodulation

Complex Envelope

The real band-pass signal $\tilde{s}(t)$ can be expressed by

$$\tilde{s}(t) = \mathrm{Re}\left\{ s(t)\, e^{j\omega_c t} \right\},$$

where ω_c and $s(t)$ denote the carrier frequency and the complex envelope respectively.

The complex envelope is given by

$$s(t) = A(t)\, e^{j\varphi(t)}$$

with $A(t)$ and $\varphi(t)$ representing an over time varying

- amplitude (amplitude modulation)
- phase (phase modulation).

Analytical Signal

Alternatively, the real signal $\tilde{s}(t)$ can be described by

$$\tilde{s}(t) = \operatorname{Re}\{s_+(t)\},$$

where $s_+(t)$ is called the analytic signal of $\tilde{s}(t)$.

The analytic signal is defined by

$$s_+(t) = \tilde{s}(t) + j\breve{\tilde{s}}(t) \quad \text{with} \quad \breve{\tilde{s}}(t) = \mathcal{H}\{\tilde{s}(t)\} = \frac{1}{\pi}\int \frac{\tilde{s}(\tau)}{t-\tau}d\tau,$$

where $\mathcal{H}\{\cdot\}$ denotes the Hilbert transform.

The Hilbert transform can be interpreted as filtering operation employing the non causal impulse response

$$\tilde{h}(t) = \frac{1}{\pi t} \quad \text{with} \quad \tilde{H}(\omega) = \mathcal{F}\left\{\frac{1}{\pi t}\right\} = -j\,\operatorname{sgn}(\omega),$$

where

$$\operatorname{sgn}(\omega)=\begin{cases}1 & \text{for } \omega>0\\ 0 & \text{for } \omega=0\\ -1 & \text{for } \omega<0\end{cases}.$$

The Fourier transform of $s_+(t)$ is given by

$$\begin{aligned} S_+(\omega) &= \tilde{S}(\omega)+j\tilde{H}(\omega)\tilde{S}(\omega) \\ &= \tilde{S}(\omega)+\operatorname{sgn}(\omega)\tilde{S}(\omega)=\begin{cases}2\tilde{S}(\omega) & \text{for } \omega>0\\ \tilde{S}(0) & \text{for } \omega=0\\ 0 & \text{for } \omega<0\end{cases}.\end{aligned}$$

Furthermore, on can show that the real and imaginary part of the analytic signal are related by

$$\operatorname{Im}\{s_+(t)\}=\frac{1}{\pi}\int\frac{\operatorname{Re}\{s_+(\tau)\}}{t-\tau}\,d\tau.$$

For narrow band signals and sufficiently large ω_c the complex envelope and the analytic signal are approximately related by

$$s(t)\,e^{j\omega_c t} \cong s_+(t).$$

For band limited signals with $B/2 < \omega_c$ we can conclude

$$s(t)\,e^{j\omega_c t} = s_+(t)$$

which implies

$$S_+(\omega) = \mathcal{F}\left\{s_+(t)\right\} = \mathcal{F}\left\{s(t)\,e^{j\omega_c t}\right\}$$

$$= S(\omega-\omega_c) = \begin{cases} 2\tilde{S}(\omega) = 2\mathcal{F}\left\{\tilde{s}(t)\right\} & \text{for } \omega > 0 \\ 0 & \text{for } \omega \le 0 \end{cases}$$

in the frequency domain.

Inphase and Quadrature Components

The real and imaginary part of the complex envelope

$$s(t) = s_I(t) + js_Q(t)$$

are called quadrature components, where $s_I(t)$ and $s_Q(t)$ denote the inphase and quadrature component respectively.

The corresponding real band-pass signal can be expressed by the inphase and quadrature components as follows.

$$\begin{aligned}\tilde{s}(t) &= \mathrm{Re}\left\{s(t)\,e^{j\omega_c t}\right\} \\ &= \mathrm{Re}\left\{\left(s_I(t) + j s_Q(t)\right)\left(\cos(\omega_c t) + j\sin(\omega_c t)\right)\right\} \\ &= s_I(t)\cos(\omega_c t) - s_Q(t)\sin(\omega_c t)\end{aligned}$$

If $\tilde{s}(t)$ is a real band-pass signal its inphase and quadrature components can be obtained by quadrature demodulation, i.e.

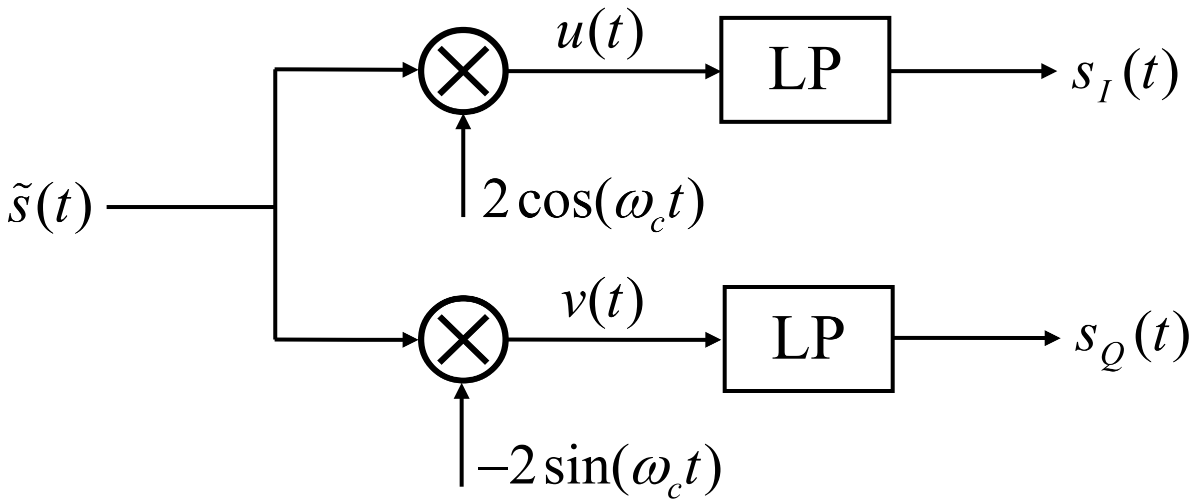
$$\begin{aligned} s_I(t) &= \mathrm{LP}\left\{2\tilde{s}(t)\cos(\omega_c t)\right\} \\ &= \mathrm{LP}\left\{s_I(t)2\cos(\omega_c t)\cos(\omega_c t) - s_Q(t)2\sin(\omega_c t)\cos(\omega_c t)\right\} \\ &= \mathrm{LP}\left\{s_I(t)\left(1+\cos(2\omega_c t)\right) - s_Q(t)\sin(2\omega_c t)\right\} \end{aligned}$$

and

$$\begin{aligned} s_Q(t) &= \mathrm{LP}\left\{-2\tilde{s}(t)\sin(\omega_c t)\right\} \\ &= \mathrm{LP}\left\{-s_I(t)2\cos(\omega_c t)\sin(\omega_c t) + s_Q(t)2\sin(\omega_c t)\sin(\omega_c t)\right\} \\ &= \mathrm{LP}\left\{-s_I(t)\sin(2\omega_c t) + s_Q(t)\left(1-\cos(2\omega_c t)\right)\right\}, \end{aligned}$$

where LP denotes the system operator of a low pass filter.

Quadrature demodulator

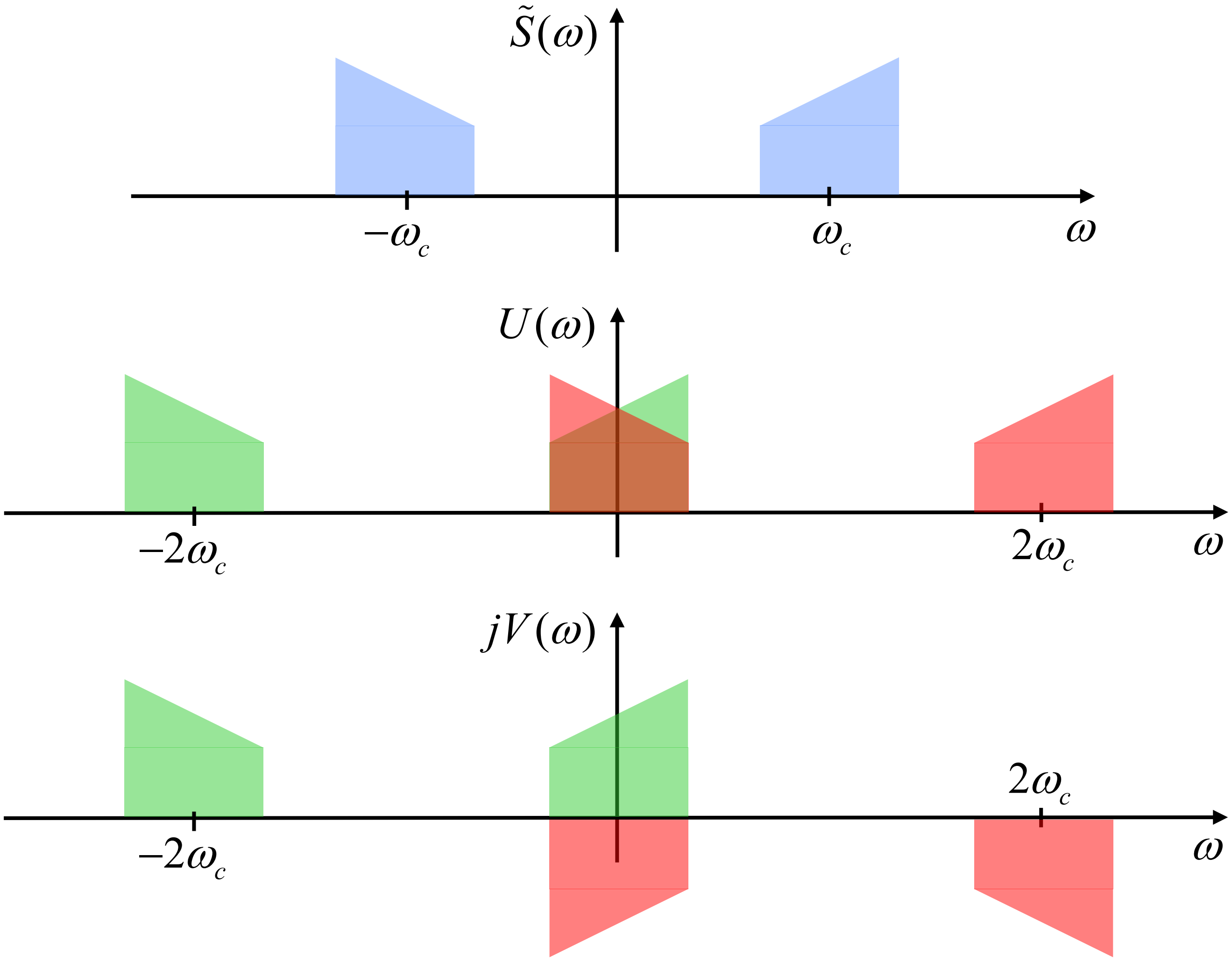


$$u(t) = 2\tilde{s}(t)\cos(\omega_c t)$$

$$= \tilde{s}(t)(e^{j\omega_c t} + e^{-j\omega_c t}) \quad \circ\!\!-\!\!\bullet \quad U(\omega) = \tilde{S}(\omega - \omega_c) + \tilde{S}(\omega + \omega_c)$$

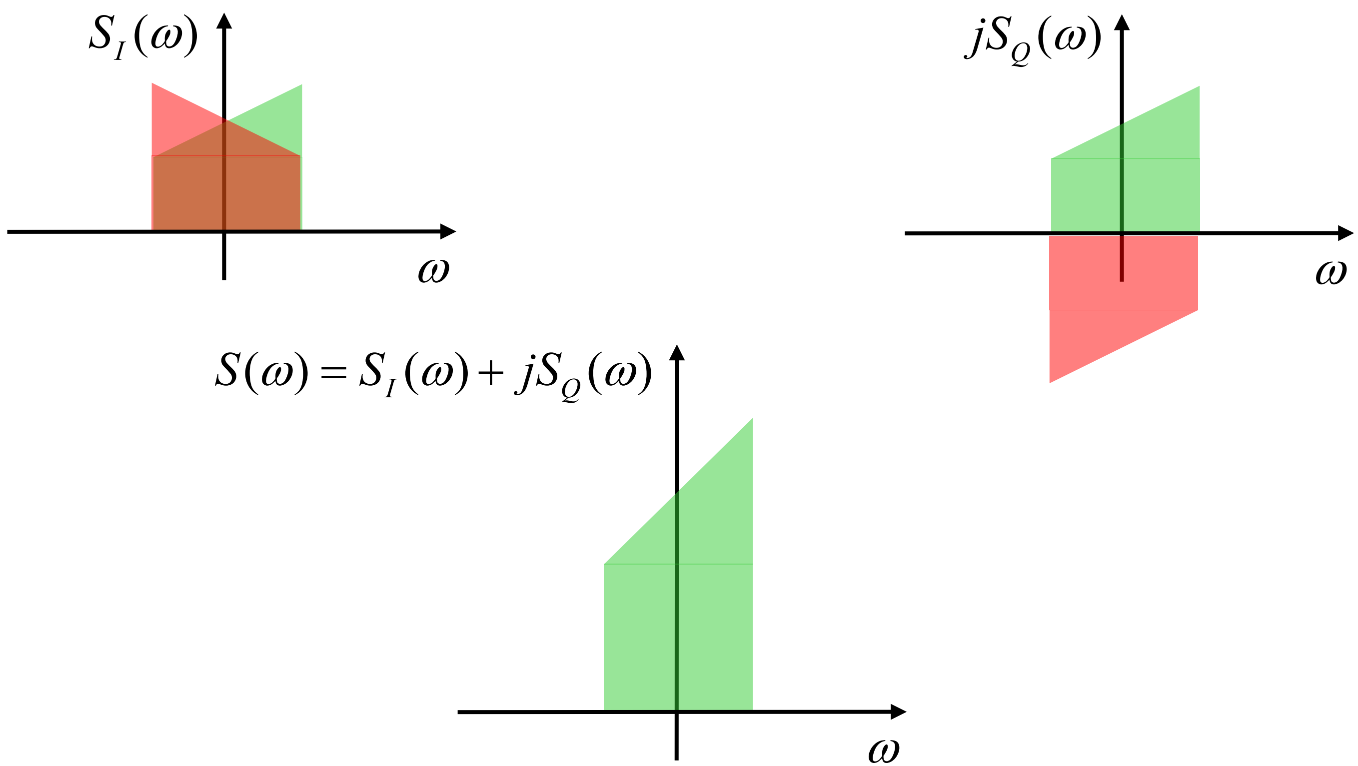
$$jv(t) = -j2\tilde{s}(t)\sin(\omega_c t)$$

$$= \tilde{s}(t)(e^{-j\omega_c t} - e^{j\omega_c t}) \quad \circ\!\!-\!\!\bullet \quad jV(\omega) = \tilde{S}(\omega + \omega_c) - \tilde{S}(\omega - \omega_c)$$


$\tilde{S}(\omega)$
$-\omega_c$
ω_c
ω
$U(\omega)$
$-2\omega_c$
$2\omega_c$
ω
$jV(\omega)$
$2\omega_c$
$-2\omega_c$
ω

$$s_I(t) = \int \tilde{h}_{LP}(\tau) u(t-\tau)\, d\tau \quad \circ\!\!-\!\!\bullet \quad S_I(\omega) = \tilde{H}_{LP}(\omega) U(\omega)$$

$$js_Q(t) = \int \tilde{h}_{LP}(\tau)\, jv(t-\tau)\, d\tau \quad \circ\!\!-\!\!\bullet \quad jS_Q(\omega) = \tilde{H}_{LP}(\omega)\, jV(\omega)$$



Complex Envelope of the noise process

The noise $\tilde{n}(t)$ is supposed to be a wide sense stationary stochastic band-pass process.

A band-pass process can be expressed by

$$\tilde{n}(t) = \mathrm{Re}\left\{ n(t)\, e^{j\omega_c t} \right\} = \frac{1}{2}\left(n(t)\, e^{j\omega_c t} + n^*(t)\, e^{-j\omega_c t} \right),$$

where $n(t)$ denotes the complex envelope.

The second order moment (correlation) function of $\tilde{n}(t)$ can be written in terms of the complex envelope as

$$r_{\tilde{n}\tilde{n}}(\tau) = \mathrm{E}\left(\tilde{n}(t+\tau)\tilde{n}(t) \right) = \frac{1}{4}\mathrm{E}\left\{ \left(n(t+\tau)\, e^{j\omega_c (t+\tau)} + n^*(t+\tau)\, e^{-j\omega_c (t+\tau)} \right) \right.$$

$$\left. \cdot \left(n(t)\, e^{j\omega_c t} + n^*(t)\, e^{-j\omega_c t} \right) \right\} =$$

$$= \frac{1}{4}\Big\{ \mathrm{E}\big(n(t+\tau)n(t)\big)e^{j\omega_c(2t+\tau)} + \mathrm{E}\big(n(t+\tau)n^*(t)\big)e^{j\omega_c\tau}$$
$$+ \mathrm{E}\big(n^*(t+\tau)n(t)\big)e^{-j\omega_c\tau} + \mathrm{E}\big(n^*(t+\tau)n^*(t)\big)e^{-j\omega_c(2t+\tau)}\Big\}.$$

Since $\tilde{n}(t)$ is assumed to be wide sense stationary, we can conclude that the equations

$$\mathrm{E}\big(n(t+\tau)n(t)\big) = 0 \quad \text{and} \quad \mathrm{E}\big(n^*(t+\tau)n^*(t)\big) = 0$$

must hold, that the second order moment (correlation) function of the complex envelope possesses the property

$$r_{nn}(\tau) = \mathrm{E}\big(n(t+\tau)n^*(t)\big)$$
$$= \Big\{\mathrm{E}\big(n^*(t+\tau)n(t)\big)\Big\}^* = \Big\{\mathrm{E}\big(n(t)n^*(t+\tau)\big)\Big\}^* = r_{nn}^*(-\tau)$$

and that consequently, the second order moment (correlation) function of $\tilde{n}(t)$ can be simplified to

$$r_{\tilde{n}\tilde{n}}(\tau) = \frac{1}{4}\left\{\mathrm{E}\left(n(t+\tau)n^*(t)\right)e^{j\omega_c\tau} + \mathrm{E}\left(n^*(t+\tau)n(t)\right)e^{-j\omega_c\tau)}\right\}$$

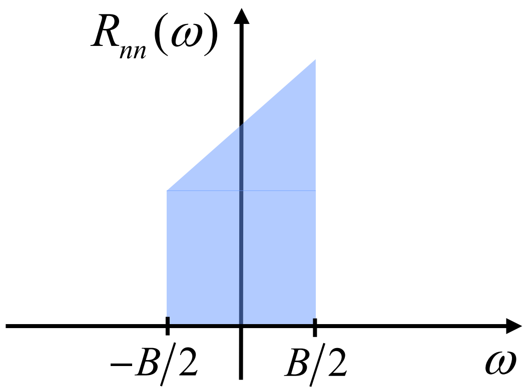
$$= \frac{1}{4}\left(r_{nn}(\tau)e^{j\omega_c\tau} + r_{nn}^*(\tau)e^{-j\omega_c\tau}\right).$$

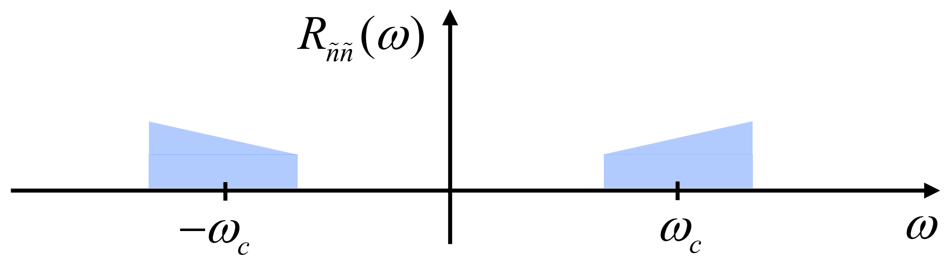
The power spectral density function of $\tilde{n}(t)$, defined by the Fourier transform of $r_{\tilde{n}\tilde{n}}(\tau)$, can be written as

$$R_{\tilde{n}\tilde{n}}(\omega) = \mathcal{F}\left\{r_{\tilde{n}\tilde{n}}(\tau)\right\} = \frac{1}{4}\left(R_{nn}(\omega-\omega_c) + R_{nn}^*(-\omega-\omega_c)\right)$$

$$= \frac{1}{4}\left(R_{nn}(\omega-\omega_c) + R_{nn}(-\omega-\omega_c)\right),$$

where $r_{nn}(\tau) = r_{nn}^*(-\tau) \circ\!\!-\!\!\bullet\, R_{nn}(\omega) = R_{nn}^*(\omega)$ has been exploited.





Finally, substituting the complex envelope of the noise, i.e. $n(t)=n_I(t)+jn_Q(t)$, in $\mathrm{E}\big(n(t+\tau)n(t)\big)=0$ the following results can be obtained.

$$\mathrm{E}\big(n(t+\tau)n(t)\big)=\mathrm{E}\big(n_I(t+\tau)n_I(t)\big)-\mathrm{E}\big(n_Q(t+\tau)n_Q(t)\big)+ j\Big\{\mathrm{E}\big(n_I(t+\tau)n_Q(t)\big)+\mathrm{E}\big(n_Q(t+\tau)n_I(t)\big)\Big\}=0$$

$$\Rightarrow\ r_{n_I n_I}(\tau)=r_{n_Q n_Q}(\tau)\ \text{ and}$$

$$r_{n_I n_Q}(\tau)=-r_{n_Q n_I}(\tau)=-r_{n_I n_Q}(-\tau)\ \Rightarrow\ r_{n_I n_Q}(0)=0$$

Signal Energy and Noise Power before and after Quadrature Demodulation

Now, we would like to investigate whether the signal energy to noise power ratio is altered by quadrature demodulation. Before quadrature demodulation the signal energy and noise power are given by

$$\|\tilde{s}\|^2 = \int_{-\infty}^{\infty} \tilde{s}^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\tilde{S}(\omega)\right|^2 d\omega = \frac{1}{\pi}\int_{0}^{\infty} \left|\tilde{S}(\omega)\right|^2 d\omega$$

and

$$\sigma_{\tilde{n}}^2 = r_{\tilde{n}\tilde{n}}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{\tilde{n}\tilde{n}}(\omega)\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} R_{\tilde{n}\tilde{n}}(\omega)\,d\omega,$$

respectively.

After quadrature demodulation we can derive

$$\|s\|^2 = \int_{-\infty}^{\infty} |s(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{0}^{\infty} |2\tilde{S}(\omega)|^2 d\omega = \frac{2}{\pi} \int_{0}^{\infty} |\tilde{S}(\omega)|^2 d\omega = 2\|\tilde{s}\|^2$$

for the signal energy and

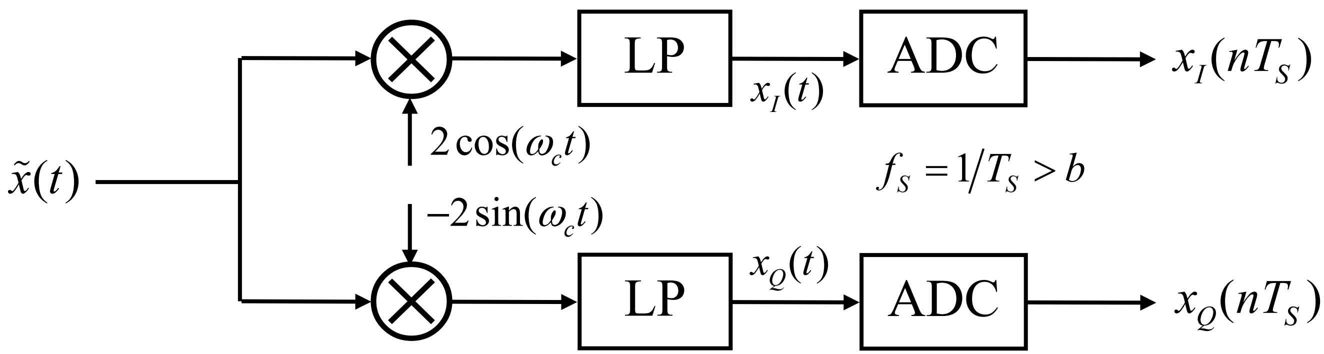
$$\sigma_n^2 = r_{nn}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{nn}(\omega)\, d\omega = \frac{2}{\pi} \int_{0}^{\infty} R_{\tilde{n}\tilde{n}}(\omega)\, d\omega = 2\sigma_{\tilde{n}}^2$$

for the noise power.

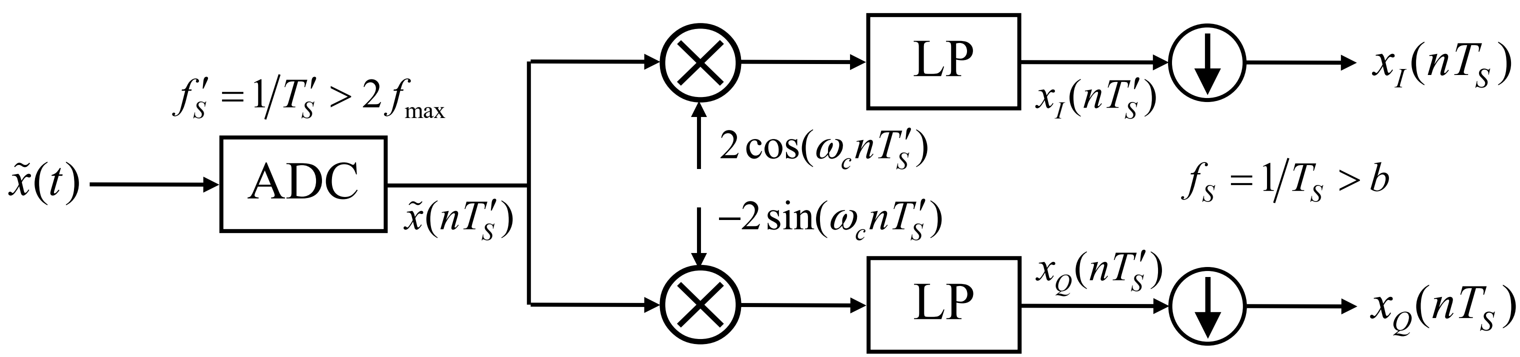
Consequently, the quadrature demodulation does not change the signal energy to noise power ratio.

Implementation Variants of Quadrature Demodulation

1) Analog Quadrature Demodulation

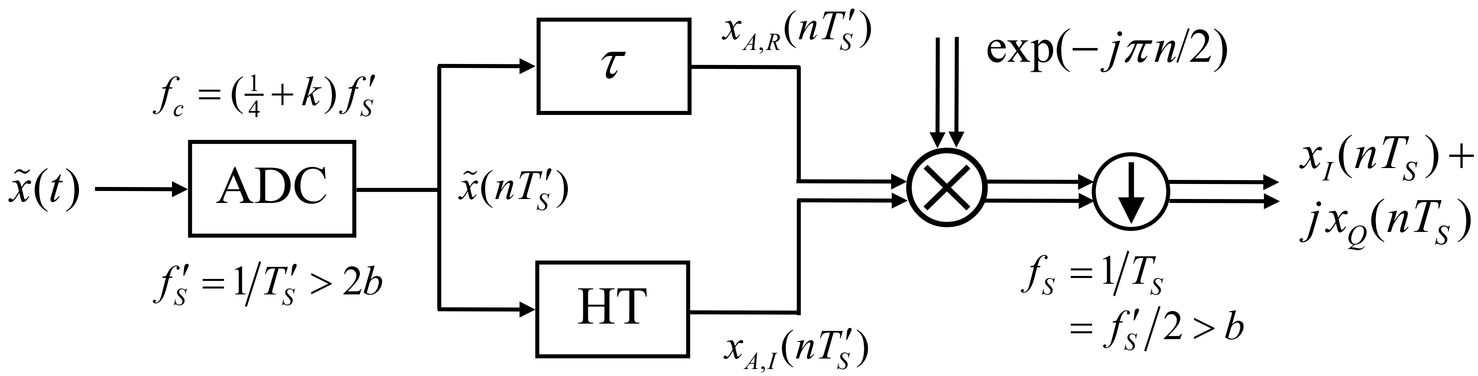


2) Digital Quadrature Demodulation

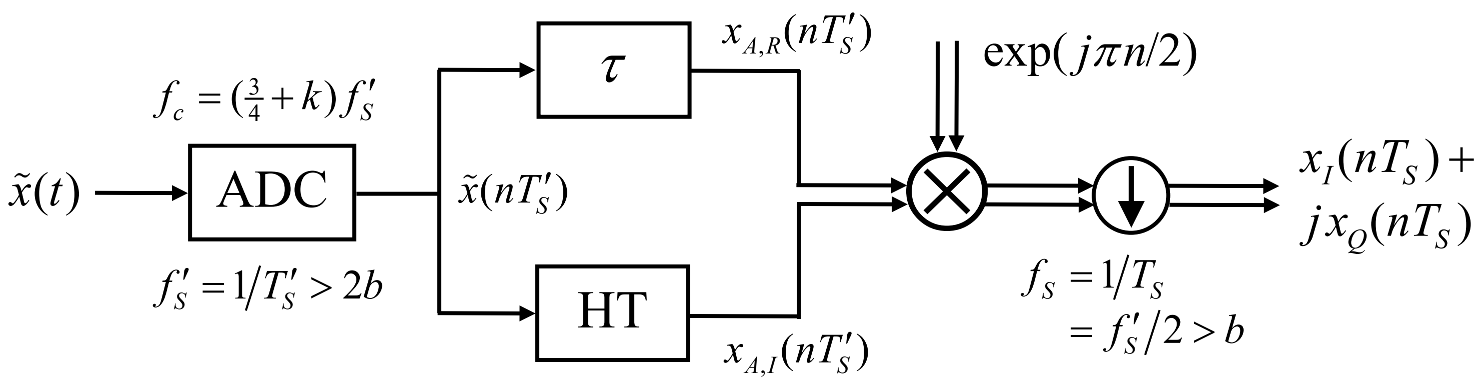


3) Digital Quadrature Demodulation with Band-Pass Sampling, Hilbert transform, complex mixing and down sampling

a)

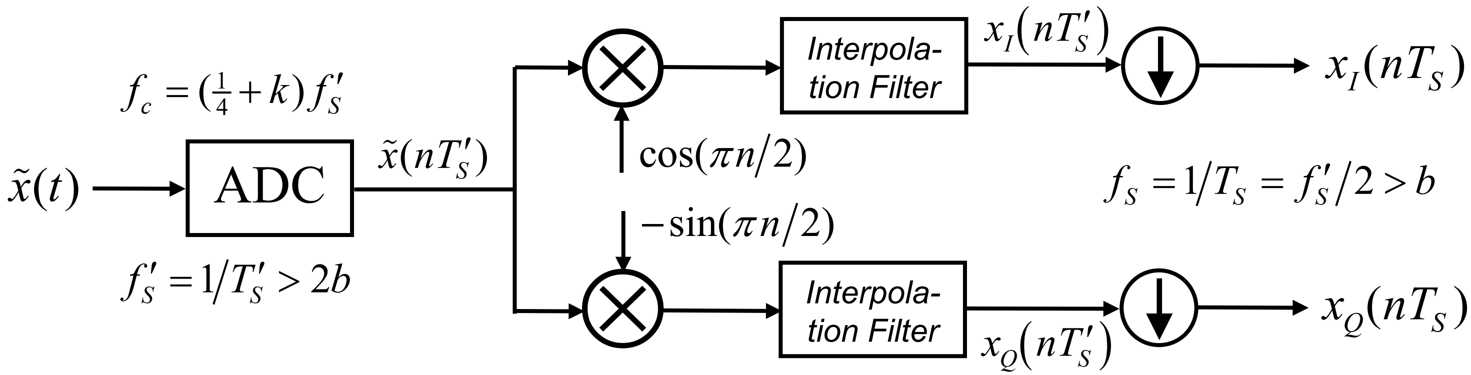


b)



4) Digital Quadrature Demodulation with Band-Pass Sampling, Real mixing, interpolation and down sampling

a)



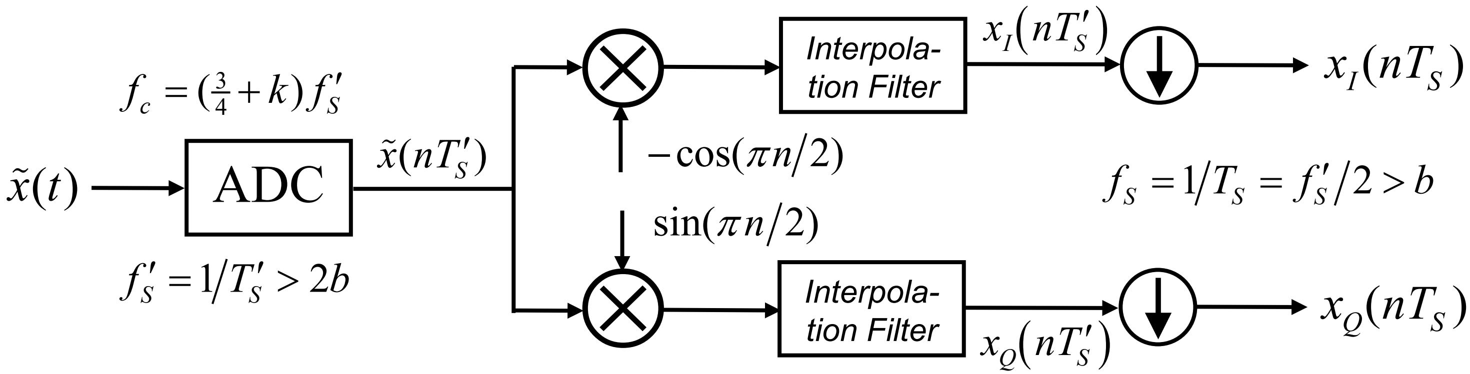
$$\tilde{x}(nT_S') = x_I(nT_S')\cos(\omega_c nT_S') - x_Q(nT_S')\sin(\omega_c nT_S')$$

$$= x_I(nT_S')\cos\left(\left(2\pi k + \frac{\pi}{2}\right)n\right) - x_Q(nT_S')\sin\left(\left(2\pi k + \frac{\pi}{2}\right)n\right)$$

$$= x_I(nT_S')\cos\left(\frac{\pi}{2}n\right) - x_Q(nT_S')\sin\left(\frac{\pi}{2}n\right)$$

Digital Quadrature Demodulation with Band-Pass Sampling, Real mixing, interpolation and down sampling (Cont.)

b)



$$\begin{aligned}\tilde{x}(nT_S') &= x_I(nT_S')\cos(\omega_c nT_S') - x_Q(nT_S')\sin(\omega_c nT_S')\\ &= x_I(nT_S')\cos\left(\left(2\pi k + \frac{3\pi}{2}\right)n\right) - x_Q(nT_S')\sin\left(\left(2\pi k + \frac{3\pi}{2}\right)n\right)\\ &= -x_I(nT_S')\cos\left(\frac{\pi}{2}n\right) + x_Q(nT_S')\sin\left(\frac{\pi}{2}n\right)\end{aligned}$$

4.2.3 Matched Filtering after Quadrature Demodulation

The transmitted and received echo signal are described by

$$\tilde{s}(t) = \mathrm{Re}\left\{ s(t)\, e^{j\omega_c t} \right\}$$

and

$$\tilde{s}_e(t) = \mathrm{Re}\left\{ s_e(t)\, e^{j\omega_c t} \right\} = \mathrm{Re}\left\{ a\, s(t-\tau)\, e^{j\omega_c (t-\tau)} \right\}$$

with $\tau = 2r/c$ denoting the travel time for a point target located in a distance r.

The complex envelop of the echo signal obtained by quadrature demodulation is given by

$$s_e(t) = a\, s(t-\tau)\, e^{-j\omega_c \tau} = a\, s(t-\tau)\, e^{-j2kr}$$

with

$$k = \omega_c / c = 2\pi / \lambda \,.$$

Thus, the complex envelope of the echo signal differs from the transmitted signal in a

- time delay $\tau = 2r/c$
- phase shift $\varphi = -2kr$
- complex constant factor a (modeling the propagation and reflection conditions for a target at location $\mathbf{r}$)

The received band-pass noise $\tilde{n}(t)$ is supposed to possess a constant spectral density over the band of interest, i.e.

$$\mathrm{R}_{\tilde{n}\tilde{n}}(\omega) = N_0/2 \ \text{ for } \ |\omega \pm \omega_c| \leq B/2.$$

Hence, the spectral density of the quadrature demodulated noise, i.e. the complex envelop $n(t)$, is determined by

$$\mathrm{R}_{nn}(\omega) = 2N_0 \ \text{ for } \ |\omega| \leq B/2.$$

Now, we want to determine the impulse response $h(t)$ of a complex valued stable receiver filter, i.e.

$$\int |h(t)|\, dt < \infty,$$

such that for the input signal

$$x(t) = s_e(t) + n(t)$$

the output signal

$$y(t) = \int h(t')x(t-t')\, dt' = \int h(t')s_e(t-t')\, dt' + \int h(t')n(t-t')\, dt'$$

possesses a maximum signal-to-noise ratio at $t = \tau$. Thus,

$$\gamma(h) = \frac{\left|\int h(t')s_e(\tau - t')\, dt'\right|^2}{\mathrm{E}\left|\int h(t')n(\tau - t')\, dt'\right|^2} = \frac{\left|s_{e,h}(\tau)\right|^2}{\mathrm{E}\left|n_h(\tau)\right|^2} = \frac{\left|s_{e,h}(\tau)\right|^2}{\sigma_{n_h}^2}$$

has to be maximized, where

$$s_{e,h}(\tau) = \int h(t')s_e(\tau - t')\,dt' = a\,e^{-j2kr}\int h(t')s(-t')\,dt'$$

and

$$n_h(\tau) = \int h(t')\,n(\tau - t')\,dt'.$$

Using Parseval's Formula the variance (power) of $n_h(t)$ can be expressed by

$$\begin{aligned}
\sigma_{n_h}^2 &= \mathrm{E}\left|n_h(t)\right|^2 = r_{n_h n_h}(0) \\
&= \frac{1}{2\pi}\int R_{n_h n_h}(\omega)\,d\omega = \frac{1}{2\pi}\int \left|H(\omega)\right|^2 R_{nn}(\omega)\,d\omega \\
&= 2N_0\,\frac{1}{2\pi}\int \left|H(\omega)\right|^2 d\omega = 2N_0\int \left|h(t)\right|^2 dt = 2N_0\left\|h\right\|^2.
\end{aligned}$$

Hence, $\gamma(h)$ can be write as

$$\gamma(h) = \frac{\left| a\, e^{-j2kr} \int h(t)s(-t)\,dt \right|^2}{2N_0 \left\| h \right\|^2} = \left|a\right|^2 \frac{\left\|s\right\|^2}{2N_0} \frac{\left| \int h(t)s(-t)\,dt \right|^2}{\left\|h\right\|^2 \left\|s\right\|^2}$$

$$= \left|a\right|^2 \frac{\left\|s\right\|^2}{2N_0} \frac{\left| \int h(t)\hat{s}^*(t)\,dt \right|^2}{\left\|h\right\|^2 \left\|\hat{s}\right\|^2}$$

with $\hat{s}^*(t) = s(-t)$ and $\left\|s\right\|^2 = \left\|\hat{s}\right\|^2$.

By means of the Cauchy Schwarz inequality

$$\left| \int f_1(t) f_2^*(t)\,dt \right|^2 \leq \int \left| f_1(t) \right|^2 dt \cdot \int \left| f_2(t) \right|^2 dt$$

one can now prove, that $\gamma(h)$ takes its maximum for

$$h_{opt}(t) = c\hat{s}(t) = cs^*(-t),$$

where, c denotes an arbitrary complex constant $\neq 0$. The maximum signal-to-noise ratio is given by

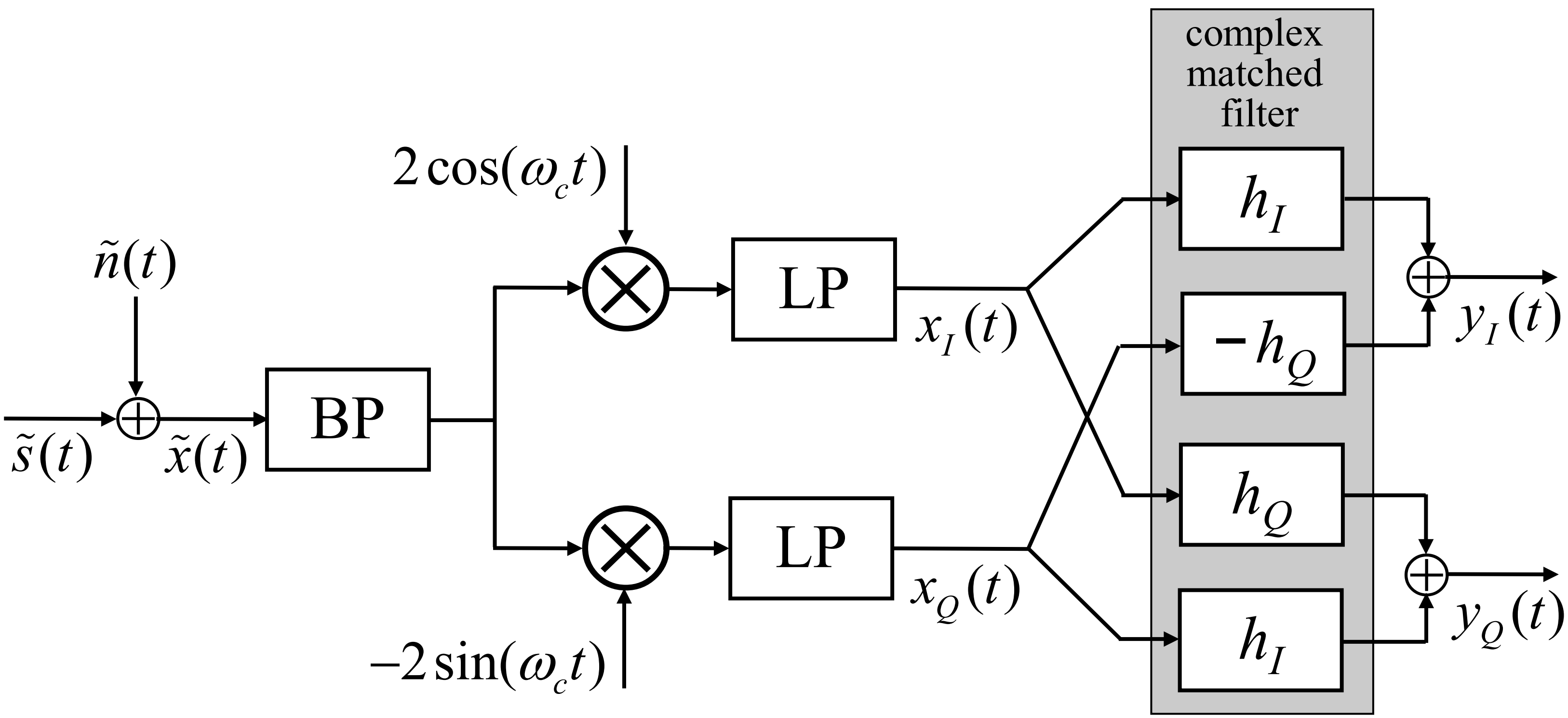
$$\gamma(h_{opt}) = |a|^2 \frac{\|s\|^2}{2N_0}.$$

If $h_{opt}(t)$ denotes the complex envelope of $\tilde{h}_{opt}(t)$ the two alternative approaches

- real filtering with $\tilde{h}_{opt}(t)$ followed by quadrature demodulation
- quadrature demodulation followed by complex filtering with $\tilde{h}_{opt}(t)$

are equivalent.

The complex filtering of quadrature demodulated signals can be implemented by real devices as depicted below.



4.3 Range Resolution of a Sonar System

A point target generates in the absence of noise the deterministic signal

$$y(t) = q(r)\, p(t-\tau) \quad \text{with} \quad \tau = 2r/c$$

at the output of the receiver, where $p(t)$ denotes the point target response, r the distance of the point target and $q(r)$ incorporates the range dependent echo amplitude and phase shift $\varphi = -2kr$.

For distributed or extended targets, we introduce the common reflectivity distribution $\tilde{a}(r)$.

Thus, due to the linearity, the output signal of the receiver filter is given by

$$\hat{a}(t) = \int \tilde{a}(r)\, q(r)\, p(t - 2r/c)\, dr,$$

i.e. the superimposition of the echoes originated along the target extend by backscattering.

Substitution of $r = c\tau/2$ in the convolution above provides

$$\hat{a}(t) = \int a(\tau)\, p(t - \tau)\, d\tau,$$

where

$$a(\tau) = \frac{c}{2}\, \tilde{a}\left(\frac{c\tau}{2}\right) q\left(\frac{c\tau}{2}\right).$$

Thus, $\hat{a}(t)$ can be understood as a reconstruction of $a(t)$ which is one of the main objectives of a sonar/radar imaging system.

A perfect reconstruction can only be achieved for

$$p(t) = \delta(t).$$

The notion range resolution shall describe a measure how far targets that provide equally strong echoes have to be separated in range to be distinguishable in the received signal.

There does not exist a unique definition for the range resolution measure.

1) Range resolution measures based on the duration of the point target response.
   a) 3 dB width: $\Delta t = t_+ - t_-$ with

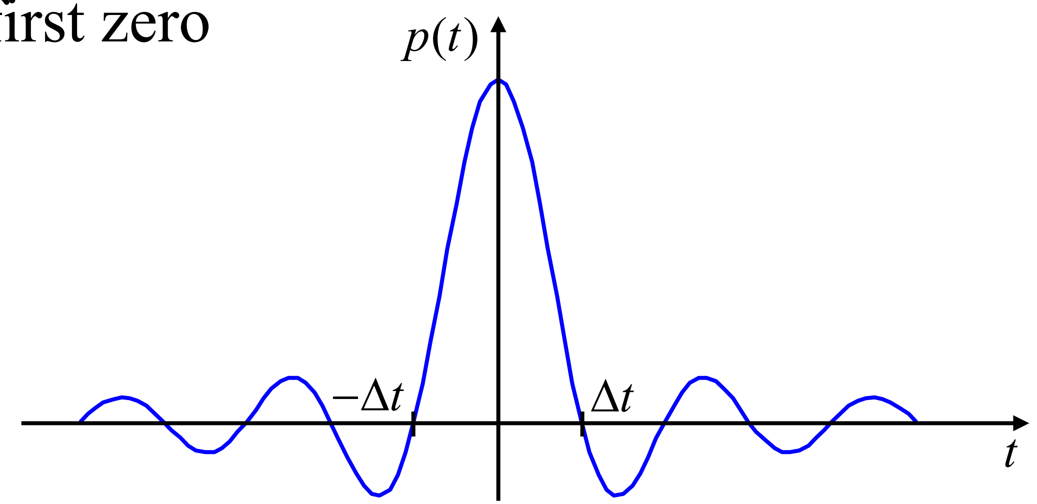
$$|p(t_-)|^2 = |p(t_+)|^2 = \frac{1}{2}|p(0)|^2 \;\Rightarrow\; \Delta r = \frac{c\Delta t}{2}$$

Example:

rectangular pulse of duration $T \Rightarrow$ triangular point target response of support $2T$ (matched filter output)

$$\Rightarrow \left(1 - \frac{\Delta t/2}{T}\right)^2 = \frac{1}{2} \Rightarrow \Delta r \cong 0.59 \frac{c}{2} T$$

b) Distance to the first zero



Example:

rectangular pulse of duration T

$\Rightarrow \;\; p(t)=0 \;\text{ for }\; |t| \ge T \;\Rightarrow\; \Delta t = T \;\Rightarrow\; \Delta r = cT/2$

c) Range resolution obtained with an energy equivalent rectangular pulse

$\|p\|^2$ = energy of the point target response $p(t)$

$\Rightarrow \;\; \|p\|^2 = |p(0)|^2 \cdot \Delta t \Rightarrow \Delta t = \|p\|^2 / |p(0)|^2$

Example:

rectangular pulse of duration T

$\Rightarrow$ triangular point target response of support $2T$

$$\Rightarrow \ \Delta t = \int_{-T}^{T} \left(1 - \frac{|t|}{T}\right)^2 dt = \ 2\int_{0}^{T} \left(1 - \frac{t}{T}\right)^2 dt$$

$$= -T\frac{2}{3}\left(1 - \frac{t}{T}\right)^3 \Bigg|_0^T = \frac{2}{3}T \ \Rightarrow \ \Delta r = \frac{2}{3}\frac{cT}{2} = \frac{cT}{3}$$

2) Resolution measure based on the separability of signals.

Two point targets generate the echo signal

$$\hat{a}(t) = a_1 p(t - \tau_1) + a_2 p(t - \tau_2),$$

where a_1 and a_2 as well as τ_1 and τ_2 denote the complex amplitudes and time delays of the echoes of target 1 and 2, respectively.

Without any loss of generality, we suppose $a_1 = 1$.

Furthermore, assuming equally strong echoes, we have

$$a_2 = e^{j\varphi}, \qquad |a_2| = 1.$$

Since φ is unknown, the worst case approach

$$f(t') = \max_{\varphi} \left| p(t'-\tau_1) + e^{j\varphi}\, p(t'-\tau_2) \right|^2$$

has to be considered. After substituting

$$t' = t + \tau_1,$$

we obtain

$$g(t) = f(t+\tau_1) = \max_{\varphi} \left| p(t) + e^{j\varphi}\, p(t-\tau) \right|,$$

where $\tau = \tau_2 - \tau_1$ denotes the echo separation in time.

Now by increasing τ from 0 towards infinity, we can observe that $g(t)$ builds up two maxima for $\tau > \Delta t$.

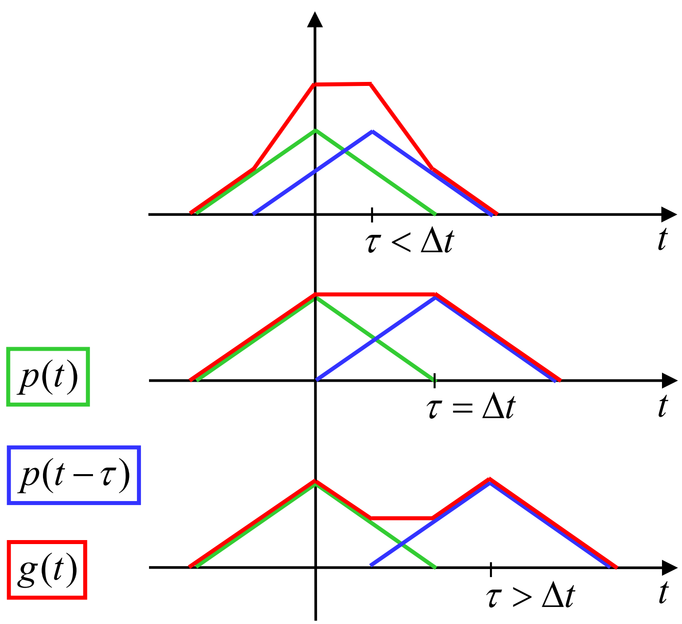
Hence, Δt can be used as a resolution measure.

Example:

rectangular pulse of duration T

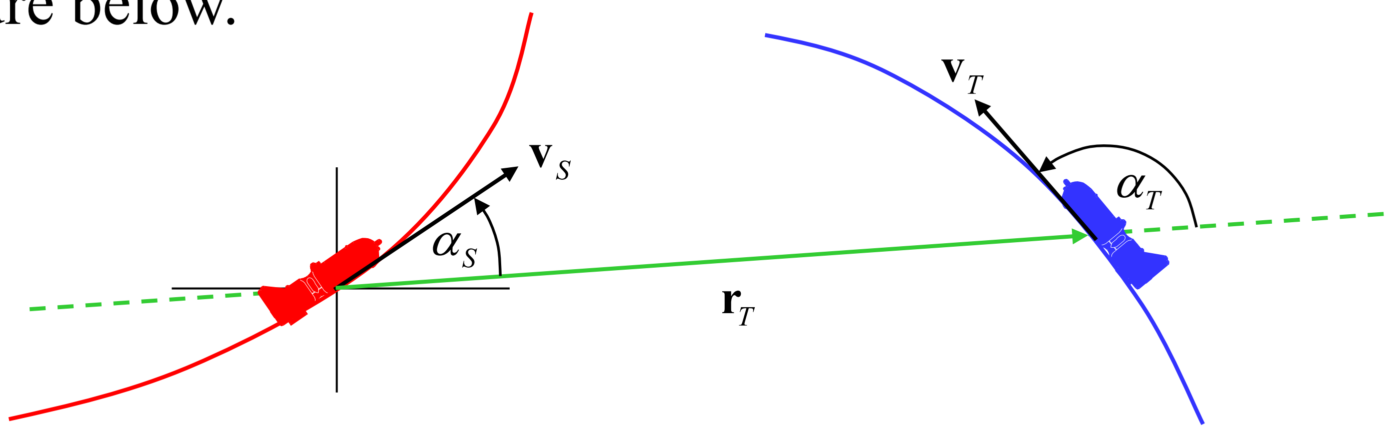
$$\Rightarrow \quad p(t) = \left(1 - |t|/T\right) \quad \text{for} \quad |t| \le T$$

$$g(t) = p(t) + p(t - \tau)$$



4.4 Doppler Effect

Moving sonar platforms as well as moving targets change the frequency of the received echo signal due to the Doppler effect. The geometry of the sonar and target motion is described in the figure below.



$f_{S,T}$ = transmitted sonar frequency $\quad f_{S,R}$ = received sonar frequency

f_T = frequency at target $\quad \tilde{v}_S = |\mathbf{v}_S| \cos\alpha_S, \quad \tilde{v}_T = |\mathbf{v}_T| \cos\alpha_T$

The frequency of the signal one would measure with a hydrophone placed on the target is given by

$$f_T = f_{S,T}\,\frac{1-\tilde{v}_T/c}{1-\tilde{v}_S/c} = f_{S,T}\,\frac{c-\tilde{v}_T}{c-\tilde{v}_S}.$$

A sound wave of this frequency is emitted/reflected by the moving target and is received by the moving sonar platform.

Hence, the frequency of the received signal is determined by

$$f_{S,R} = f_T\,\frac{1+\tilde{v}_S/c}{1+\tilde{v}_T/c} = f_T\,\frac{c+\tilde{v}_S}{c+\tilde{v}_T} = f_{S,T}\,\frac{(c-\tilde{v}_T)(c+\tilde{v}_S)}{(c-\tilde{v}_S)(c+\tilde{v}_T)}$$

$$= f_{S,T}\,\frac{\left(1-\tilde{v}_T/c\right)\left(1+\tilde{v}_S/c\right)}{\left(1-\tilde{v}_S/c\right)\left(1+\tilde{v}_T/c\right)} = f_{S,T}\,\frac{1-(\tilde{v}_T-\tilde{v}_S)/c-\tilde{v}_T\tilde{v}_S/c^2}{1+(\tilde{v}_T-\tilde{v}_S)/c-\tilde{v}_T\tilde{v}_S/c^2}.$$

After supposing $\tilde{v}_T \tilde{v}_S \ll c^2$, we obtain

$$f_{S,R} \approx f_{S,T} \frac{1-(\tilde{v}_T - \tilde{v}_S)/c}{1+(\tilde{v}_T - \tilde{v}_S)/c} = f_{S,T} \frac{1-v_r/c}{1+v_r/c} = f_{S,T} \frac{c-v_r}{c+v_r},$$

where

$$v_r = \tilde{v}_T - \tilde{v}_S$$

denotes the relative radial speed between the sonar platform and the target.

<u>Remark:</u> For Radar (electromagnetic waves) holds

$$f_T = f_{R,T} \sqrt{\frac{c-v_r}{c+v_r}} \Rightarrow f_{R,R} = f_{R,T} \frac{c-v_r}{c+v_r}$$

$f_{R,T}$ = transmitted radar frequency $f_{R,R}$ = received radar frequency

f_T = frequency at target c = speed of light

Example:

Supposing $\tilde{v}_T = -10\ \mathrm{m/s},\ \tilde{v}_S = 5\ \mathrm{m/s} \Rightarrow \tilde{v}_r = -15\ \mathrm{m/s}$

$c = 1500\ \mathrm{m/s}$,

we obtain

$$f_{S,R} = 1.02020157\, f_{S,T} \quad \text{(exact calculation)}$$

$$\approx 1.02020202\, f_{S,T} \quad \text{(approximative calculation)}$$

$$\Rightarrow \text{ relative error} < 5 \cdot 10^{-7}.$$

For the subsequent considerations we suppose that only the target is moving, i.e.

$$f_{S,R} = f_{S,T} \frac{1 - \tilde{v}_T / c}{1 + \tilde{v}_T / c} = f_{S,T} \frac{c - \tilde{v}_T}{c + \tilde{v}_T}.$$

Thus, the distance between the sonar platform and the moving target can be expressed as a function of time by

$$r(t) = r_0 + \tilde{v}_T(t)\,t.$$

The signal received at time t was reflected by the target at time

$$t' = t - \tau(t)/2,$$

where $\tau(t)$ denotes the two-way travel time of the signal. Consequently, $\tau(t)$ is only implicitly expressed by

$$\tau(t) = 2\,r(t')/c = 2\,r\big(t - \tau(t)/2\big)/c.$$

The signal received (real band-pass signal) is therefore

$$\tilde{s}_e(t) = \mathrm{Re}\left\{a\,s\big(t-\tau(t)\big)\,e^{j\omega_c(t-\tau(t))}\right\} \text{ with } \tilde{s}(t) = \mathrm{Re}\left\{s(t)\,e^{j\omega_c t}\right\},$$

where $s(t)$ is the complex envelope of the transmitted signal.

Assuming now

$$\tilde{v}_T(t) = \tilde{v}_T = const.$$

we obtain

$$\tau(t) = \frac{2}{c}\left(r_0 + \tilde{v}_T\left(t - \tau(t)/2\right)\right) = \frac{2}{c}(r_0 + \tilde{v}_T t) - \frac{\tilde{v}_T}{c}\tau(t)$$

which after some reformulations, i.e.

$$\tau(t)\left(1 + \tilde{v}_T/c\right) = \frac{2}{c}(r_0 + \tilde{v}_T t)$$

and

$$\tau(t) = \frac{2}{c}\frac{r_0 + \tilde{v}_T t}{1 + \tilde{v}_T/c} = 2\frac{r_0 + \tilde{v}_T t}{c + \tilde{v}_T},$$

leads us to the expression

$$t-\tau(t)=\frac{(c+\tilde{v}_T)t-2(r_0+\tilde{v}_T t)}{c+\tilde{v}_T}=\frac{(c-\tilde{v}_T)t-2r_0}{c+\tilde{v}_T}$$

$$=\frac{c-\tilde{v}_T}{c+\tilde{v}_T}\left(t-\frac{2r_0}{c-\tilde{v}_T}\right)=\alpha(t-\tilde{\tau}_0)$$

with

$$\alpha=\frac{c-\tilde{v}_T}{c+\tilde{v}_T},\quad \tau_0=\frac{2r_0}{c}\ \text{ and }\ \tilde{\tau}_0=\frac{2r_0}{c-\tilde{v}_T}=\tau_0\frac{c}{c-\tilde{v}_T}=\tau_0\frac{1}{1-\tilde{v}_T/c}.$$

The received signal can be expressed by

$$\tilde{s}_e(t)=\mathrm{Re}\left\{a\,s\big(\alpha(t-\tilde{\tau}_0)\big)e^{j\omega_c\alpha(t-\tilde{\tau}_0)}\right\}$$

$$=\mathrm{Re}\left\{a\,s\big(\alpha(t-\tilde{\tau}_0)\big)e^{-j\omega_c\alpha\tilde{\tau}_0}e^{j\omega_c(\alpha-1)t}e^{j\omega_c t}\right\}=\mathrm{Re}\left\{s_e(t)\,e^{j\omega_c t}\right\}.$$

Thus, the complex envelope is given by

$$s_e(t) = a\,s\big(\alpha(t-\tilde{\tau}_0)\big)\,e^{-j\omega_c\alpha\tilde{\tau}_0}\,e^{j\omega_c(\alpha-1)t}.$$

The impacts[1)] caused by the Doppler effect are

1) Alteration of frequency

$$\omega = \omega_c\alpha = \omega_c + \omega_c(\alpha-1) = \omega_c + \omega_{dop}$$

with $\omega_{dop} = (\alpha-1)\omega_c$

2) Time dilatation of the complex envelope by the factor α

3) Alteration of time delay by the factor

$$1\big/\left(1-\tilde{v}_T/c\right)$$

[1)] ordered with respect to importance

The impacts can be approximately considered as follows.

1) $$\omega_{dop} = (\alpha - 1)\omega_c = \left(\frac{c - \tilde{v}_T}{c + \tilde{v}_T} - 1 \right)\omega_c$$

$$= \frac{-2\tilde{v}_T}{c + \tilde{v}_T}\omega_c = -\frac{2\tilde{v}_T}{c}\omega_c \frac{c}{c + \tilde{v}_T} = -\frac{2\tilde{v}_T}{c}\omega_c \frac{1}{1 + \tilde{v}_T / c}$$

Example:

$$\left. \begin{array}{l} \tilde{v}_T = 15 \text{ m/s} \\ c = 1500 \text{ m/s} \end{array} \right\} \Rightarrow \frac{1}{1 + \tilde{v}_T / c} = \frac{1}{1 + 1/100} = 0.\overline{9900} \approx 1$$

$$\Rightarrow \omega_{dop} \approx -\frac{2\tilde{v}_T}{c}\omega_c$$

2) The difference between τ_0 and $\tilde{\tau}_0$
   a) can be neglected with regard to the time shift of the complex envelope.
   b) can not be neglected with regard to the phase shift provided by $\exp\left(-j\omega_c\alpha\tilde{\tau}_0\right)$.
   However, since the initial phase is usually unknown in practice the impact of the phase shift does not require additional attention.

3) Time dilatation of the complex envelope reduces the performance of matched filtering (correlation).
   Its impact can be neglected if the phase shift for $f_{\max}$ satisfies

$$2\pi f_{\max}\left|\alpha T - T\right| \ll \pi \;\Rightarrow\; bT \ll \left|\frac{1}{\alpha - 1}\right| \approx \left|\frac{c}{2v_r}\right|$$

with

$f_{\max} = \text{maximum frequency} = b/2$ $\quad T = \text{pulse length}$

$b = \text{bandwidth} = B/2\pi$ $\quad bT = \text{time bandwidth product.}$

Example:

$$\tilde{v}_T = 2.5\,\text{m/s},\;\; c = 1500\,\text{m/s} \;\Rightarrow\; bT \ll \left|\frac{c}{2\tilde{v}_T}\right| = 300$$

After applying the three approximations, we finally can write

$$s_e(t) \approx a\,s(t-\tau_0)\,e^{j\omega_{dop}(t-\tau_0)}\,e^{-j\omega_c\tau_0}.$$

4.5 Pulse Compression

The range resolution and signal energy are determined by

$$\Delta r \cong cT/2 \text{ and } \|s\|^2 = PT,$$

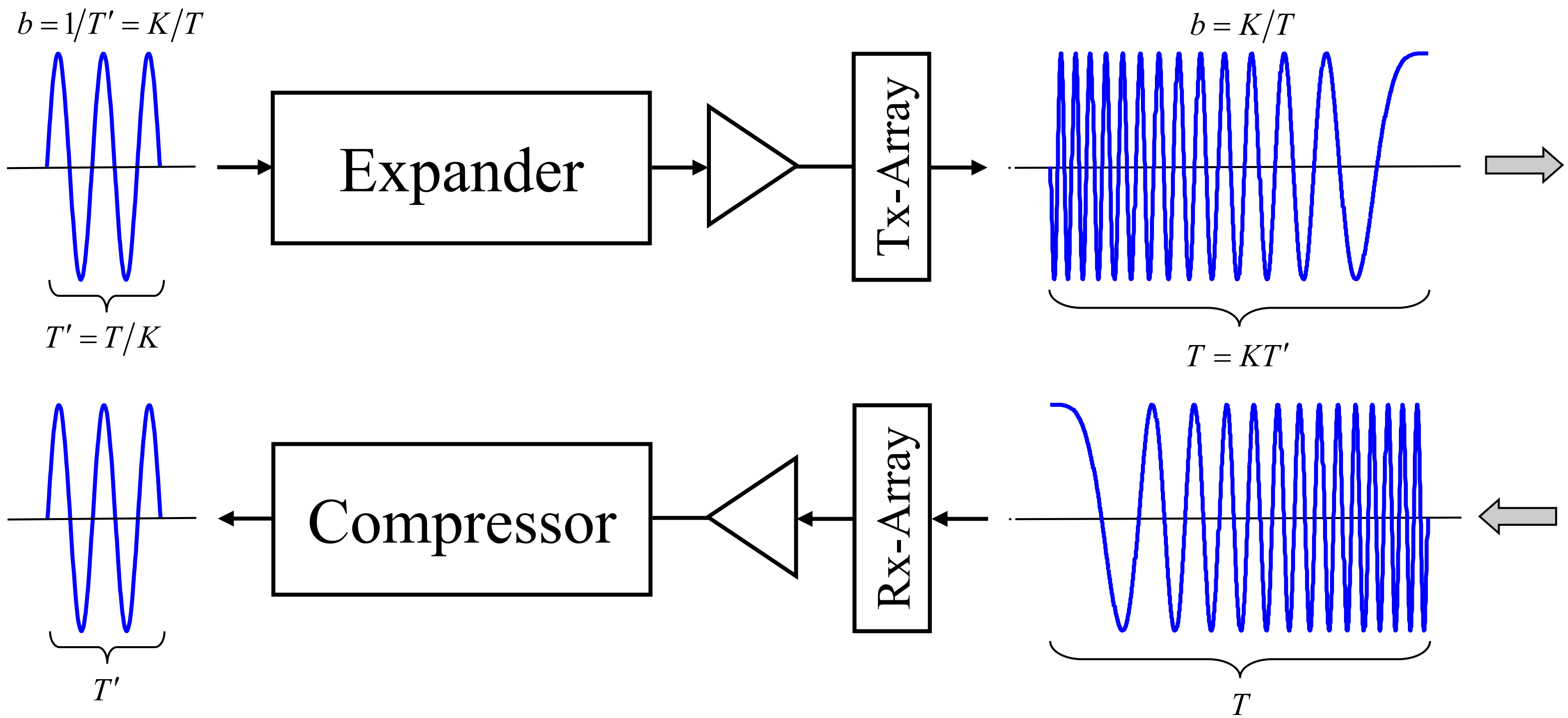
where c, T and P denote the sound speed, pulse length and transmitting power, respectively.

The power P is technically/physically limited by the capabilities of the power amplifiers and the power dependent occurrence of cavitation at the transducers radiation surface.

The retention of signal energy ($\propto SNR$) and the enhancement of range resolution seem to be contradicting goals.

Therefore, how can the range resolution be enhanced without losing signal energy for a given maximum transmitting power?

Heuristic Solution



Pulse expansion and compression, e.g. via a dispersive delay line, where K denotes the so-called compression factor.

Transmitting energy: $\|s\|^2 = PT$

Range resolution: $\Delta r \cong \frac{c}{2}\frac{T}{K} = \frac{c}{2}T'$

Time-Bandwidth-Product: $bT = \begin{cases} 1 & \text{for rect pulse} \\ K & \text{for expanded pulse} \end{cases}$

Hence, the compression factor coincides with the time band-width product.

The range resolution is determined by the bandwidth

$$\Delta r \cong \frac{c}{2}T' = \frac{c}{2b}.$$

4.5.1 Interconnection of power spectrum, point target response and range resolution

The range resolution is given by

$$\Delta r = c\Delta t/2,$$

where Δt indicates the time extent of the point target response

$$p(t) = r_{ss}(t) = \int s(t+\tau)\, s^*(\tau)\, d\tau = \int s(\tau)\, \underbrace{s^*(-t+\tau)}_{h_{opt}(t-\tau)}\, d\tau$$

which is equivalent to the autocorrelation function.

For the autocorrelation function holds

$$r_{ss}(t) = \mathcal{F}^{-1}\left\{\left|S(\omega)\right|^2\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|S(\omega)\right|^2 e^{j\omega t}\, d\omega.$$

Example:

A signal with power spectrum

$$|S(\omega)|^2 = 1_{(-\pi b,\pi b)}(\omega)$$

possesses the point target response

$$r_{ss}(t) = b\frac{\sin(\pi bt)}{\pi bt} = b\,\mathrm{si}(\pi bt) \Rightarrow r_{ss}\left(\frac{1}{b}\right) = 0 \Rightarrow \Delta t_{\substack{\text{distance}\\\text{to first zero}}} = \frac{1}{b}.$$

Furthermore, approximately holds

$$r_{ss}\left(-\frac{1}{2b}\right) = r_{ss}\left(+\frac{1}{2b}\right) \cong \frac{r_{ss}(0)}{\sqrt{2}} \Rightarrow \Delta t_{3dB} \cong \frac{1}{b}$$

and for large b more precisely

$$\Delta t_{3dB} \cong 0.88/b.$$

Remarks:

- The point target response/autocorrelation function is completely determined by the power spectrum of the signal.
- The bandwidth of the signal determines the range resolution.

4.5.2 Ambiguity function

The ambiguity function is defined by

$$\chi(\tau,\nu) := \int_{-\infty}^{\infty} s(t)\, s^*(t-\tau)\, e^{j2\pi\nu t}\, dt.$$

It can be interpreted as the output of a matched filter designed for a Doppler frequency shift f_0 if a signal with Doppler frequency shift $f_0+\nu$ is received.

Thus, $\chi(\tau,\nu)$ can be understood as the point target response in the Range/Doppler domain.

$$\chi(\tau,\nu)=\int_{-\infty}^{\infty} s(t)\,s^*(t-\tau)\,e^{j2\pi\nu t}\,dt$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} S(\omega)\,e^{j\omega t}\,\frac{d\omega}{2\pi}\right)\left(\int_{-\infty}^{\infty} S^*(\omega')\,e^{-j\omega'(t-\tau)}\,\frac{d\omega'}{2\pi}\right)e^{j2\pi\nu t}\,dt$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\omega)\,S^*(\omega')\,e^{j\omega'\tau}\,e^{j(\omega-\omega'+2\pi\nu)t}\,d\omega\,d\omega'\,dt$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(\omega)\,S^*(\omega')\,e^{j\omega'\tau}\,\delta(\omega-\omega'+2\pi\nu)\,d\omega\,d\omega'$$

$$=\frac{1}{4\pi^2}\int_{-\infty}^{\infty} S(\omega'-2\pi\nu)\,S^*(\omega')\,e^{j\omega'\tau}\,d\omega'$$

Ambiguity-Function of particular waveforms

a) Rectangular pulse

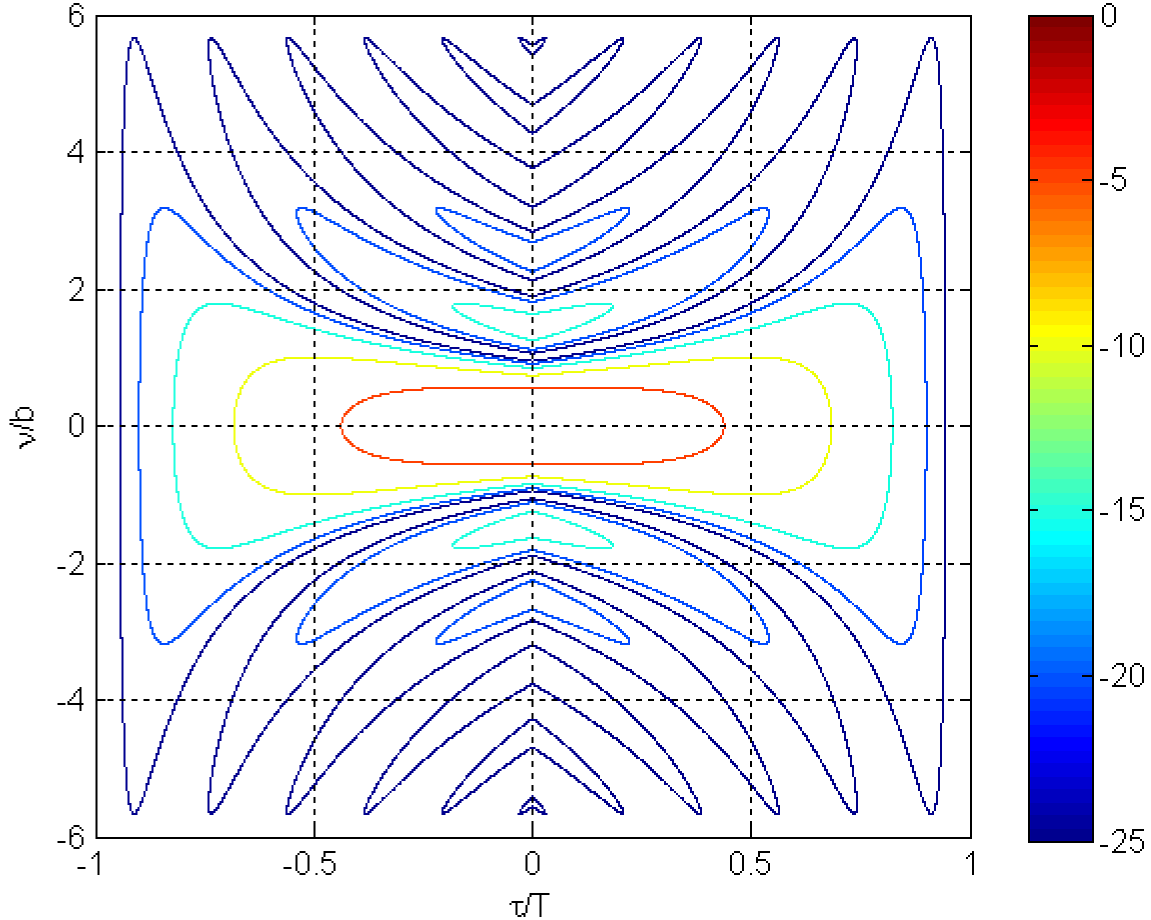
$$s(t) = \frac{1}{\sqrt{T}} 1_{(-T/2,T/2)}(t)$$

with $\|s\|^2 = 1$. Hence, the ambiguity function is given by

$$\chi(\tau,\nu) = \int_{-\infty}^{\infty} s(t)\, s^*(t-\tau)\, e^{j2\pi\nu t}\, dt$$

$$= \begin{cases} e^{j\pi\nu\tau}\left(1-\frac{|\tau|}{T}\right)\frac{\sin\left(\pi\nu(T-|\tau|)\right)}{\pi\nu(T-|\tau|)} & \text{for } |\tau| \le T \\ 0 & \text{elsewhere} \end{cases}.$$

Ambiguity function of a rectangular pulse



b) LFM pulse with rectangular envelope

$$s(t) = \frac{1}{\sqrt{T}} 1_{(-T/2,T/2)}(t)\exp\left(j\pi k t^2\right)$$

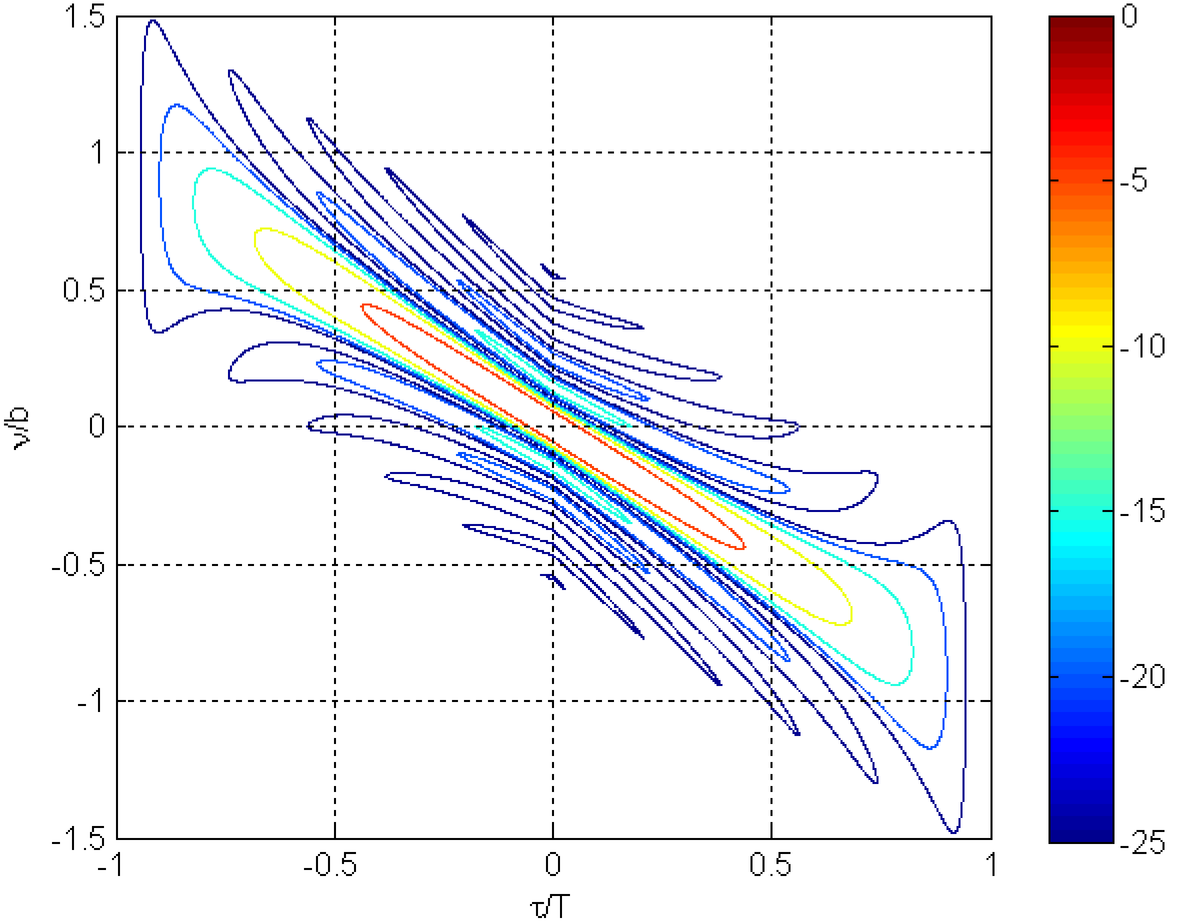
with $|k| = b/T$.

In this case the ambiguity function can be expressed by

$$\chi(\tau,\nu) = \int_{-\infty}^{\infty} s(t)\, s^*(t-\tau)\, e^{j2\pi\nu t}\, dt$$

$$= \begin{cases} e^{j\pi\nu\tau}\left(1-\frac{|\tau|}{T}\right)\frac{\sin\left(\pi(k\tau+\nu)(T-|\tau|)\right)}{\pi(k\tau+\nu)(T-|\tau|)} & \text{for } |\tau| \le T \\ 0 & \text{elsewhere} \end{cases}$$

Ambiguity function of a LFM pulse with rectangular envelope



c) LFM pulse with Gaussian envelope

$$s(t)=\frac{1}{\sqrt[4]{\pi\sigma^2}}\exp\left(-\frac{t^2}{2\sigma^2}+j\pi k t^2\right),$$

where σ (standard deviation) and the effective pulse duration T are related by

$$T=\sqrt{2\pi}\,\sigma$$

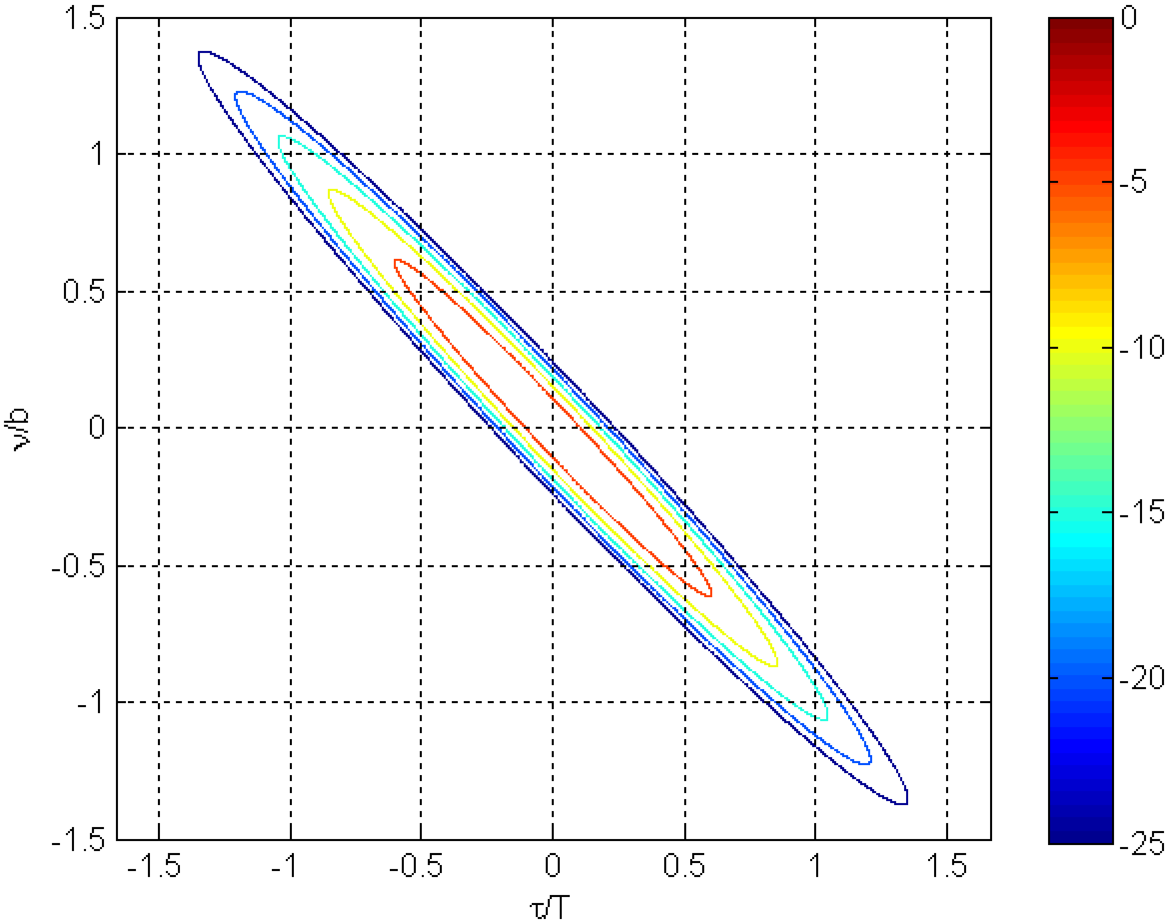
and where k determines the slope of the LFM with $|k|=b/T$.

After some manipulations, we obtain

$$\chi(\tau,\nu)=\int_{-\infty}^{\infty} s(t)\,s^*(t-\tau)\,e^{j2\pi\nu t}\,dt$$

$$=e^{j\pi\nu\tau}\exp\left(-\tau^2/(4\sigma^2)-\pi\,\sigma^2(k\tau+\nu)^2\right).$$

Ambiguity function of a LFM pulse with Gaussian envelope



Assignment 8:

1) Show, that the ambiguity function of an LFM pulse with rectangular envelope can be expressed as given on p. 75.
2) Develop a Matlab program for determining the
   - spectra of the waveforms a) – c), using the FFT
   - ambiguity functions given in a) – c) in analytical form

Invariance of the Volume under the ambiguity surface

The following calculations show that the volume under the ambiguity surface does not depend on the waveform. The volume depends only on the signal energy.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\chi(\tau,\nu)|^2 \, d\tau d\nu =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)\,e^{j2\pi\nu t} s^*(t')s(t'-\tau)\,e^{-j2\pi\nu t'} dt\,dt'd\tau\,d\nu$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)s^*(t')s(t'-\tau)\delta(t-t')\,dt\,dt'd\tau$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(t)s^*(t-\tau)s^*(t)s(t-\tau)\,dt\,d\tau$$

$$= \int_{-\infty}^{\infty} |s(t)|^2 \left(\int_{-\infty}^{\infty} |s(t-\tau)|^2 \, d\tau \right) dt.$$

After substituting

$$t' = t - \tau \quad \text{with} \quad dt' = -d\tau,$$

we obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\chi(\tau,\nu)|^2 \, d\tau d\nu = \int_{-\infty}^{\infty} |s(t)|^2 \int_{-\infty}^{\infty} |s(t')|^2 \, dt' dt$$

$$= \int_{-\infty}^{\infty} |s(t)|^2 \, dt \int_{-\infty}^{\infty} |s(t')|^2 \, dt'$$

$$= \left(\int_{-\infty}^{\infty} |s(t)|^2 \, dt \right)^2 = \|s\|^4 = |\chi(0,0)|^2 .$$

4.6 Signal Detection

In the signal detection theory for sonar applications the following cases are distinguished:

1) The signal is completely known.
2) The amplitude of the signal is known and the phase is modeled as an uniformly distributed random variable.
3) The amplitude and phase of the signal are modeled as a Rayleigh and an uniformly distributed random variable, respectively.

Furthermore, assuming white and normally distributed noise all cases lead to optimum detectors that mainly base on a matched filter approach.

Since the detectors exploit test statistics with different statistical distributional properties they clearly do not possess the same detection capabilities.

For instance, a detector assuming 2) requires in comparison with a detector utilizing 1) an increased signal-to-noise ratio (*SNR*) of approximately 1 dB.

Nevertheless, common to these detectors is that the performance can be parameterized by the *SNR* of the matched filter output.

The received signal can be described in discrete-time by

$$x(lT_S) = s_e(lT_S) + n(lT_S)$$

with

$$s_e(lT_S) = \eta\, s(lT_S - \tau),$$

where s, s_e and n denote the transmitted signal, the echo signal and the noise, and where η and τ describe the propagation/target scattering loss and the two-way travel time, respectively.

Supposing the noise variance σ_n^2 to be known, we exemplarily solve case 2) of the aforementioned sonar target detection problems by the following hypothesis test using the notation

$$\mathbf{x}_l = \left(x\left(lT_S\right),\ldots,x\left((l+K-1)T_S\right)\right)^T$$

$$\mathbf{n}_l = \left(n\left(lT_S\right),\ldots,n\left((l+K-1)T_S\right)\right)^T$$

$$\mathbf{s} = \left(s\left(0\right),\ldots,s\left((K-1)T_S\right)\right)^T \quad \text{with} \quad l = 0,1,\ldots \quad \text{and} \quad K = \lfloor T/T_S \rfloor.$$

Hypothesis Testing

1) Setting up of a hypothesis H_0

   $\mathbf{x}_l$ does not contain the signal waveform $\mathbf{s}$, i.e.

$$H_0: \quad \mathbf{x}_l = \mathbf{n}_l, \quad \mathbf{x}_l \sim \mathcal{CN}_K(\mathbf{0}, \sigma_n^2 \mathbf{I})$$

2) Setting up of an alternative H_1

   $\mathbf{x}_l$ contains the signal waveform $\mathbf{s}$, i.e.

$$H_1: \quad \mathbf{x}_l = \eta \mathbf{s} + \mathbf{n}_l, \quad \mathbf{x}_l \sim \mathcal{CN}_K(\eta \mathbf{s}, \sigma_n^2 \mathbf{I})$$

$$\text{with} \quad \eta = \tilde{\eta} e^{j\varphi}, \ \tilde{\eta}, \varphi \in \mathbb{R},$$

where $\tilde{\eta} > 0$ and φ the on $[-\pi, \pi)$ uniformly distributed phase of the echo signal.

3) The statistic $t(\mathbf{x}_l)$ of the observation $\mathbf{x}_l$ for testing the hypothesis H_0 is given by the normalized magnitude of the matched filter output, i.e.

$$t(\mathbf{x}_l) = \frac{\left|\sum_{k=0}^{K-1} s(kT_S)^* x\big((l+k)T_S\big)\right|}{\sqrt{E_T \sigma_n^2}} = \frac{\left|\mathbf{s}^H \mathbf{x}_l\right|}{\sqrt{E_T \sigma_n^2}},$$

where $E_T = \mathbf{s}^H \mathbf{s}$ denotes the transmitted signal energy.

4) Determination of the probability density function of the statistic $T = t(\mathbf{x}_l)$ under H_0 provides the Rayleigh density

$$f_T(t \mid H_0) = \begin{cases} 2t \exp(-t^2) & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

5) Calculation of the threshold for discarding hypothesis H_0. For a given probability of false alarm P_{FA} the threshold κ can be determined as follows.

$$P_{FA} = P\left(T > \kappa \,|\, H_0\right)$$

$$= 2\int_{\kappa}^{\infty} t \exp(-t^2)\, dt = \exp(-\kappa^2) \Rightarrow \kappa = \sqrt{-\ln(P_{FA})}$$

6) If $t(\mathbf{x}_l) > \kappa$ one decides for H_1, i.e. $\mathbf{x}_l$ contains the waveform $\mathbf{s}$, with $P_{FA} = \alpha$. If $t(\mathbf{x}_l) \le \kappa$ one decides for H_0, i.e. $\mathbf{x}_l$ does not contain the waveform $\mathbf{s}$.

7) Determination of the probability density function of the statistic $T = t(\mathbf{x}_l)$ under H_1 provides the Rice density

$$f_T(t \mid H_1) = 2t \exp\left(-t^2 - \frac{E_E}{\sigma_n^2}\right) I_0\left(2\sqrt{\frac{E_E}{\sigma_n^2}}\, t\right),$$

where $E_E = (\eta \mathbf{s})^H (\eta \mathbf{s})$ denotes the energy of the echo signal and where

$$I_0(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(x \cos \vartheta)\, d\vartheta$$

is the Bessel-function of the first kind and order zero.

8) Calculation of the probability of detection P_D.

For a given threshold κ the probability of detection can be determined by

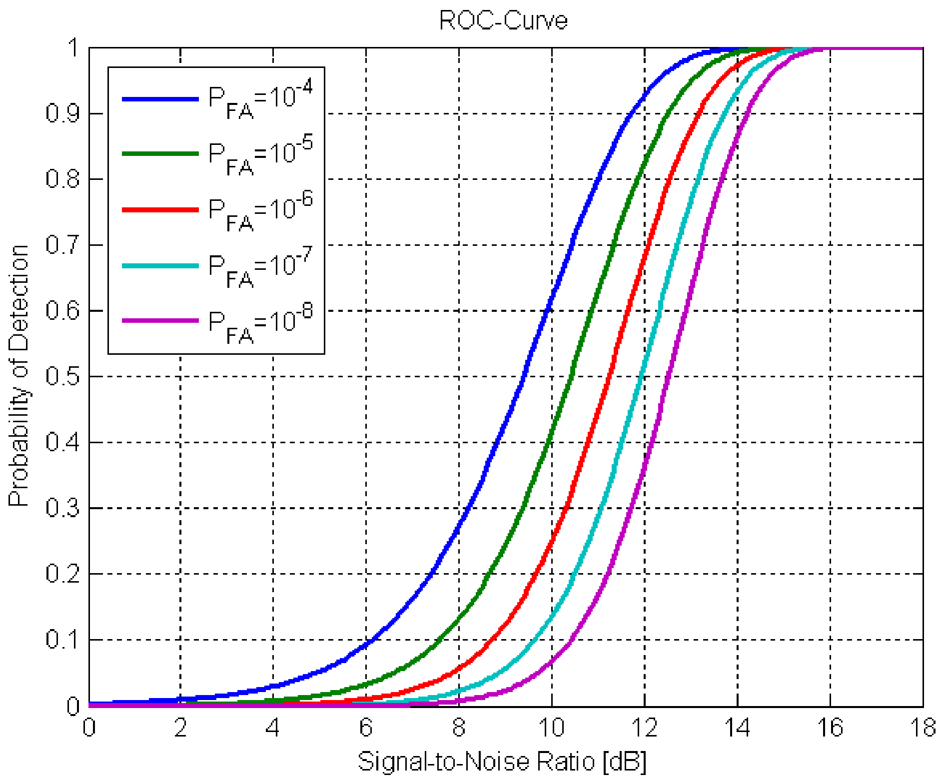
$$P_D = P\left(T > \kappa \mid H_1\right)$$
$$= \int_{\kappa}^{\infty} 2t \exp\left(-t^2 - \frac{E_E}{\sigma_n^2}\right) I_0\left(2\sqrt{\frac{E_E}{\sigma_n^2}}\, t\right) dt$$
$$= \int_{\tilde{\kappa}}^{\infty} z \exp\left(-\frac{1}{2}\left(z^2 + \frac{2E_E}{\sigma_n^2}\right)\right) I_0\left(\sqrt{\frac{2E_E}{\sigma_n^2}}\, z\right) dz = Q(d, \tilde{\kappa})$$

with $d^2 = 2\frac{E_E}{\sigma_n^2} = 2\left(\frac{S}{N}\right)_{out}$ and $\tilde{\kappa} = \sqrt{2}\,\kappa = \sqrt{-2\ln(P_{FA})}$,

where Q is the so-called Marcum's Q-function

$$Q(\alpha, \beta) = \int_{\beta}^{\infty} z \exp\left(-\frac{1}{2}\left(z^2 + \alpha^2\right)\right) I_0\left(\alpha z\right) dz.$$

9) The evaluation of the P_D as a function of SNR and parameterized by various P_{FA} provides the ROC-Curves depicted in the following figure.



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