Underwater Acoustics and Sonar Signal Processing

Contents

5 Array Processing

5.1 Introduction

The sensors spatially sample the wave field at the locations $\mathbf{r}_n$ for $n = 1,2,...,N$. This yields a set of complex signals (complex envelopes or analytical signals) which we collect in the vector

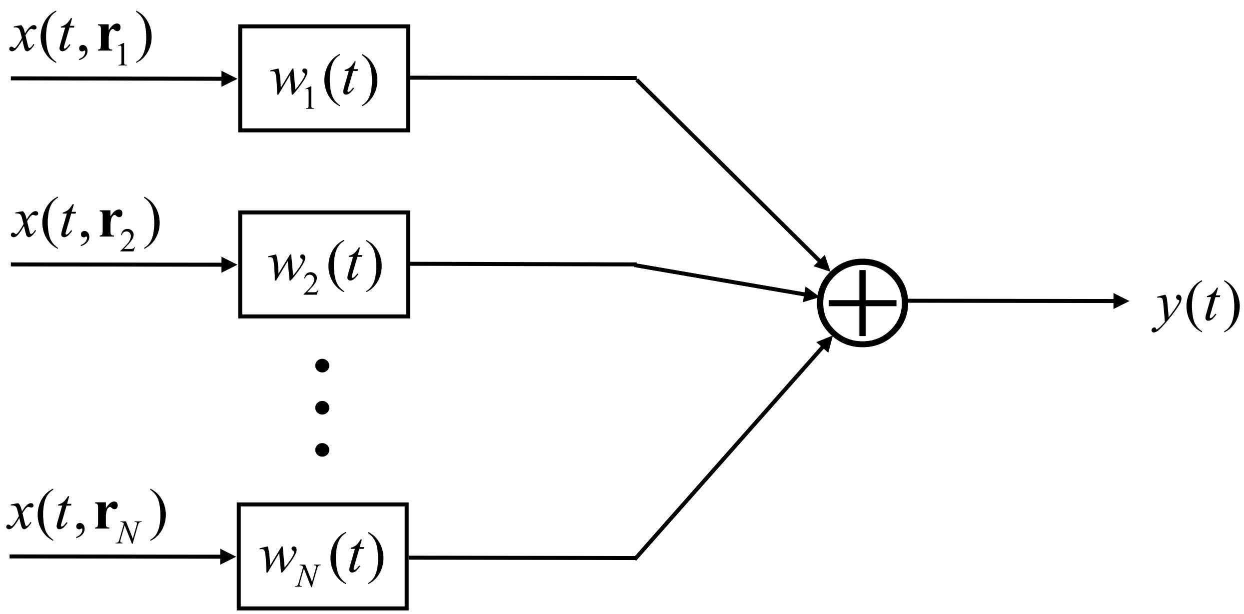
$$\mathbf{x}(t) = \left(x_1(t),\ldots,x_N(t)\right)^T = \left(x(t,\mathbf{r}_1),\ldots,x(t,\mathbf{r}_N)\right)^T .$$

We process each complex sensor output signal by a linear time invariant filter with complex impulse response $w_n(t)$ and summing up the outputs to obtain the complex array output signal

$$y(t) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} w_n(t-\tau)\, x(\tau,\mathbf{r}_n)\, d\tau$$

or in vector notation

$$y(t)=\int_{-\infty}^{\infty}\mathbf{w}^T(t-\tau)\,\mathbf{x}(\tau)\,d\tau \quad \text{with} \quad \mathbf{w}(t)=\big(w_1(t),\ldots,w_N(t)\big)^T .$$



In the Frequency domain we can obtain

$$Y(\omega) = \mathcal{F}\{y(t)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} \mathbf{w}^T(t-\tau)\,\mathbf{x}(\tau)\,d\tau\right\} = \mathbf{W}^T(\omega)\,\mathbf{X}(\omega),$$

where

$$\mathbf{X}(\omega) = \int_{-\infty}^{\infty} \mathbf{x}(t)\,e^{-j\omega t}dt \quad \text{and} \quad \mathbf{W}(\omega) = \int_{-\infty}^{\infty} \mathbf{w}(t)\,e^{-j\omega t}dt.$$

If $s(t)$ is the signal that would be received at the origin of the coordinate system, i.e. at $\mathbf{r} = \mathbf{0}$, we can write

$$\begin{aligned}\mathbf{x}(t) = \left(x_1(t),\ldots,x_N(t)\right)^T &= \left(x(t,\mathbf{r}_1), x(t,\mathbf{r}_2),\ldots,x(t,\mathbf{r}_N)\right)^T \\ &= \left(s(t-\tau_1), s(t-\tau_2),\ldots,s(t-\tau_n)\right)^T,\end{aligned}$$

where

$$\tau_n = -\mathbf{k}^T \mathbf{r}_n / \omega \quad ^{1)}$$

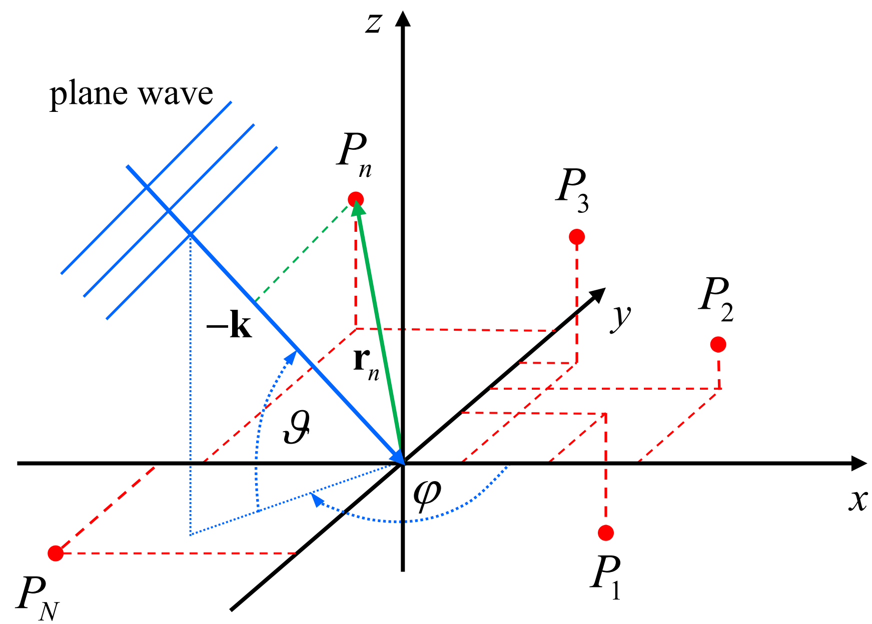
with

$$\mathbf{r}_n = (x_n, y_n, z_n)^T = \overrightarrow{OP_n}, \quad n = 1, \ldots, N$$

and

$$\mathbf{k} = k \begin{pmatrix} \cos\varphi \cos\vartheta \\ \sin\varphi \cos\vartheta \\ \sin\vartheta \end{pmatrix},$$

$$k = \omega / c = 2\pi / \lambda.$$



[1)] Minus sign arises because the plan wave propagates in direction of $-\mathbf{k}$.

The time delays can be expressed by

$$\tau_n = -(x_n \cos\varphi \cos\vartheta + y_n \sin\varphi \cos\vartheta + z_n \sin\vartheta)/c\,.$$

Defining direction cosines with respect to each axis, i.e.

$$\xi_x = \cos\varphi \cos\vartheta,\ \ \xi_y = \sin\varphi \cos\vartheta,\ \ \xi_z = \sin\vartheta$$

with $\boldsymbol{\xi} = (\xi_x, \xi_y, \xi_z)^T$, we can write

$$\tau_n = -(x_n \xi_x + y_n \xi_y + z_n \xi_z)/c = -\boldsymbol{\xi}^T \mathbf{r}_n / c\,.$$

The Fourier transform of $x(t, \mathbf{r}_n) = s(t - \tau_n)$ leads to

$$X(\omega, \mathbf{r}_n) = \int_{-\infty}^{\infty} x(t, \mathbf{r}_n)\, e^{-j\omega t} dt = \int_{-\infty}^{\infty} s(t - \tau_n)\, e^{-j\omega t} dt$$

$$= e^{-j\omega\tau_n} S(\omega) = e^{j\mathbf{k}^T \mathbf{r}_n} S(\omega).$$

Defining

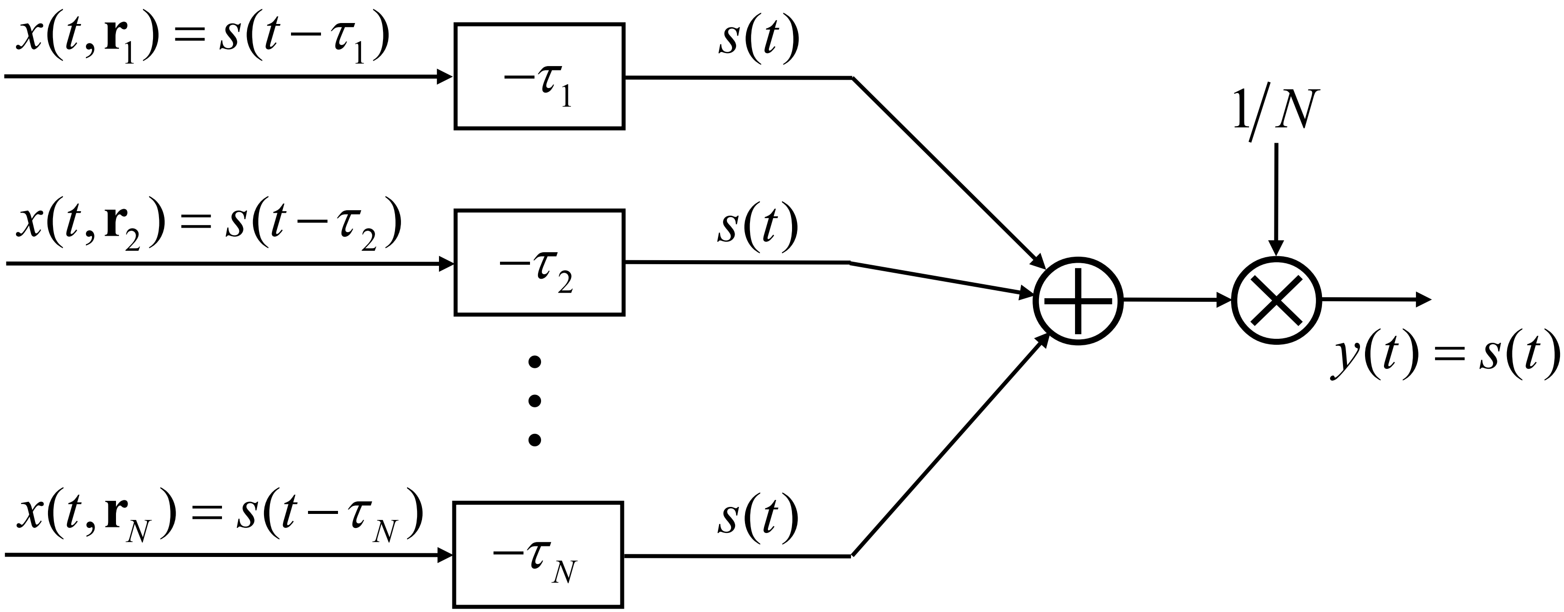
$$\mathbf{a}(\mathbf{k}) = \left(\exp(j\mathbf{k}^T\mathbf{r}_1),\ldots,\exp(j\mathbf{k}^T\mathbf{r}_N)\right)^T$$

we can write

$$\mathbf{X}(\omega) = \begin{pmatrix} X_1(\omega) \\ \vdots \\ X_N(\omega) \end{pmatrix} = \begin{pmatrix} X(\omega,\mathbf{r}_1) \\ \vdots \\ X(\omega,\mathbf{r}_N) \end{pmatrix} = S(\omega)\,\mathbf{a}(\mathbf{k}).$$

The vector **a**(**k**) incorporates all of the spatial characteristics of the array and is referred to the array manifold vector.

Now, the array output signal is formed by adding the sensor signals after they have been aligned by time shifts.



Hence, $w_n(t) = \dfrac{1}{N}\delta(t+\tau_n),\ \ n = 1,\ldots,N.$

This processor is referred to as a delay-and-sum beamformer or conventional beamformer.

In practice one adds a common delay τ in each channel so that the operations are physically realizable, i.e.

$$-\tau_n + \tau \geq 0, \quad \forall n = 1, \ldots, N.$$

If $\mathbf{k}_s$ denotes the wave vector of a plane wave of interest, i.e.

$$\tau_n = -\frac{1}{\omega}\mathbf{k}_s^T \mathbf{r}_n = -\frac{1}{c}\boldsymbol{\xi}_s^T \mathbf{r}_n,$$

the Fourier transform of $\mathbf{w}(t) = \left(w_1(t), \ldots, w_N(t)\right)^T$ provides

$$\mathbf{W}(\omega) = \int_{-\infty}^{\infty} \mathbf{w}(t)\, e^{-j\omega t} dt = \frac{1}{N}\mathbf{a}(-\mathbf{k}_s),$$

where

$$\mathbf{W}^T(\omega) = \frac{1}{N}\mathbf{a}^T(-\mathbf{k}_s) = \frac{1}{N}\mathbf{a}^H(\mathbf{k}_s).$$

Generally, we want to find the response of the array $y(t)$ to an input wave field $x(t,\mathbf{r})$.

The systems theory approach of analyzing the response of a linear time invariant system in terms of the superposition of complex exponential basis functions can be extended to space-time signals.

The basis functions are now plane waves of the form

$$x(t,\mathbf{r}) = A\,e^{j(\omega t + \mathbf{k}^T \mathbf{r})}.$$

Spatial sampling at the locations $\mathbf{r}_n$, $n = 1,\ldots,N$ provides the signal vector

$$\mathbf{x}(t) = \left(x_1(t),\ldots,x_N(t)\right)^T = \left(x(t,\mathbf{r}_1),\ldots,x(t,\mathbf{r}_N)\right)^T = A\,e^{j\omega t}\,\mathbf{a}(\mathbf{k})$$

The plane wave response of the conventional beamformer is

$$y(t) = \mathbf{W}^T(\omega)\,\mathbf{a}(\mathbf{k})\,A\,e^{j\omega t},$$

where the temporal spatial processing is completely described by the so-called frequency-wave number response function

$$b(\omega,\mathbf{k}) = \mathbf{W}^T(\omega)\,\mathbf{a}(\mathbf{k})$$

of the array. Replacing in $b(\omega,\mathbf{k})$ the wave vector by

$$\mathbf{k} = \frac{\omega}{c}\left(\cos\varphi\cos\vartheta, \sin\varphi\cos\vartheta, \sin\vartheta\right)^T = \frac{\omega}{c}\boldsymbol{\xi}(\varphi,\vartheta),$$

where c is assumed to be known, we obtain the beam pattern

$$\tilde{b}(\omega,\varphi,\vartheta) = b(\omega,\mathbf{k})\Big|_{\mathbf{k}=\frac{\omega}{c}\boldsymbol{\xi}(\varphi,\vartheta)}.$$

Example:

Supposing

$$\mathbf{W}^T(\omega) = \frac{1}{N}\mathbf{a}^H(\mathbf{k}_s) \text{ with } \mathbf{k}_s = \frac{\omega}{c}\boldsymbol{\xi}_s = \frac{\omega}{c}\boldsymbol{\xi}(\varphi_s, \vartheta_s)$$

we obtain

$$\begin{aligned}\tilde{b}(\omega,\varphi,\vartheta) &= \frac{1}{N}\mathbf{a}^H\left(\frac{\omega}{c}\boldsymbol{\xi}(\varphi_s,\vartheta_s)\right)\mathbf{a}\left(\frac{\omega}{c}\boldsymbol{\xi}(\varphi,\vartheta)\right) \\ &= \frac{1}{N}\sum_{n=1}^{N}\exp\left\{ j\frac{\omega}{c}\Big[x_n\big(\xi_x(\varphi,\vartheta)-\xi_x(\varphi_s,\vartheta_s)\big)\right. \\ &\qquad +y_n\big(\xi_y(\varphi,\vartheta)-\xi_y(\varphi_s,\vartheta_s)\big) \\ &\qquad \left. +z_n\big(\xi_z(\varphi,\vartheta)-\xi_z(\varphi_s,\vartheta_s)\big)\Big]\right\}\end{aligned}$$

a) linear vertical array, i.e. $x_n = 0, y_n = 0$ for $n = 1,\ldots,N$

$$\tilde{b}(\omega,\varphi,\vartheta) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(j\frac{\omega}{c} z_n(\sin\vartheta - \sin\vartheta_s)\right)$$

with $z_n = \left(n - (N+1)/2\right)d,\ n = 1,\ldots,N$ we obtain

$$\tilde{b}(\omega,\varphi,\vartheta) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(j\frac{\omega}{c}\left(n - \frac{N+1}{2}\right)d(\sin\vartheta - \sin\vartheta_s)\right)$$

and with $d = \lambda/2,\ \omega/c = 2\pi/\lambda$ holds

$$\tilde{b}(\varphi,\vartheta) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(j\left(n - \frac{N+1}{2}\right)\pi(\sin\vartheta - \sin\vartheta_s)\right)$$

b) linear horizontal array, i.e. $x_n = 0$, $z_n = 0$ for $n = 1,\dots,N$

$$\tilde{b}(\omega,\varphi,\vartheta) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(j\frac{\omega}{c}y_n(\sin\varphi\cos\vartheta - \sin\varphi_s\cos\vartheta_s)\right)$$

for $\vartheta = \vartheta_s = 0 \;\Rightarrow\; \tilde{b}(\omega,\varphi,0) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(j\frac{\omega}{c}y_n(\sin\varphi - \sin\varphi_s)\right)$

with $y_n = \left(n-(N+1)/2\right)d$, $n = 1,\dots,N$ we obtain

$$\tilde{b}(\omega,\varphi,0) = \frac{1}{N}\sum_{n=1}^{N}\exp\left(j\frac{\omega}{c}\left(n-\frac{N+1}{2}\right)d(\sin\varphi - \sin\varphi_s)\right)$$

and with $d = \lambda/2$, $\omega/c = 2\pi/\lambda$ holds

$$\tilde{b}(\varphi,0)=\frac{1}{N}\sum_{n=1}^{N}\exp\left(j\left(n-\frac{N+1}{2}\right)\pi(\sin\varphi-\sin\varphi_s)\right)$$

c) planar array in yz-plane, i.e. $x_n = 0$ for $n = 1,\ldots,N$

$$\tilde{b}(\omega,\varphi,\vartheta)=\frac{1}{N}\sum_{n=1}^{N}\exp\left\{j\frac{\omega}{c}\left[y_n(\sin\varphi\cos\vartheta-\sin\varphi_s\cos\vartheta_s)+z_n(\sin\vartheta-\sin\vartheta_s)\right]\right\}$$

with $N = KL,\ \ y_k=\left(k-(K+1)/2\right)d_y,\ \ k=1,\ldots,K$

$$z_l=\left(l-(L+1)/2\right)d_z,\quad l=1,\ldots,L$$

we obtain

$$\tilde{b}(\omega,\varphi,\vartheta)=\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\exp\left\{j\frac{\omega}{c}\Big[\big(k-(K+1)/2\big)d_y(\sin\varphi\cos\vartheta-\sin\varphi_s\cos\vartheta_s)+\big(l-(L+1)/2\big)d_z(\sin\vartheta-\sin\vartheta_s)\Big]\right\}$$

and with $d_y=d_z=\lambda/2,\ \omega/c=2\pi/\lambda$ holds

$$\tilde{b}(\omega,\varphi,\vartheta)=\frac{1}{N}\sum_{k=1}^{K}\sum_{l=1}^{L}\exp\left\{j\pi\Big[\big(k-(K+1)/2\big)(\sin\varphi\cos\vartheta-\sin\varphi_s\cos\vartheta_s)+\big(l-(L+1)/2\big)(\sin\vartheta-\sin\vartheta_s)\Big]\right\}$$

5.2 Performance Measures

To quantify the gain in signal-to-noise ratio obtained by beam-forming the following figures of merit are of interest.

5.2.1 Array Gain

Let $N(\omega,\varphi,\vartheta)$ denote the frequency-angular spectrum of the noise field and $\tilde{b}(\omega,\varphi,\vartheta)$ the beam pattern of an array of om-nidirectional hydrophones. Then the frequency dependent array gain $AG(\omega)$ is defined as

$$AG(\omega)=10\log_{10}\left(\frac{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}N(\omega,\varphi,\vartheta)\cos\vartheta\,d\vartheta\,d\varphi}{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}N(\omega,\varphi,\vartheta)\left|\tilde{b}(\omega,\varphi,\vartheta)\right|^{2}\cos\vartheta\,d\vartheta\,d\varphi}\right).$$

5.2.2 Directivity Index

In case that $N(\omega,\varphi,\vartheta)$ represents the frequency-angular spectrum of an spatially isotropic noise field, i.e.

$$N(\omega,\varphi,\vartheta)=\tilde{N}(\omega),$$

the array gain $AG(\omega)$ simplifies to the so-called directivity index $DI(\omega)$ given by

$$DI(\omega)=10\log_{10}\left(\frac{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\tilde{N}(\omega)\cos\vartheta\,d\vartheta\,d\varphi}{\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\tilde{N}(\omega)\left|\tilde{b}(\omega,\varphi,\vartheta)\right|^2\cos\vartheta\,d\vartheta\,d\varphi}\right)$$

$$=10\log_{10}\left(4\pi\Big/\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\left|\tilde{b}(\omega,\varphi,\vartheta)\right|^2\cos\vartheta\,d\vartheta\,d\varphi\right).$$

Example:

For an $\lambda/2$ equidistantly spaced linear vertical array with

$$\tilde{b}(\varphi,\vartheta)=\frac{1}{N}\sum_{n=1}^{N} e^{j\left(n-\frac{N+1}{2}\right)\pi(\sin\vartheta-\sin\vartheta_s)}=\bar{b}(\vartheta)$$

we obtain

$$DI=10\log_{10}\left(4\pi\Big/\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\left|\bar{b}(\vartheta)\right|^2\cos\vartheta\,d\vartheta\,d\varphi\right)=10\log_{10}(N),$$

where

$$\int_{-\pi}^{\pi}\int_{-\pi/2}^{\pi/2}\left|\bar{b}(\vartheta)\right|^2\cos\vartheta\,d\vartheta\,d\varphi=2\pi\int_{-\pi/2}^{\pi/2}\left|\bar{b}(\vartheta)\right|^2\cos\vartheta\,d\vartheta=$$

$$= \frac{2\pi}{N^2} \int_{-\pi/2}^{\pi/2} \sum_{n=1}^{N} e^{j\left(n - \frac{N+1}{2}\right)\pi(\sin\vartheta - \sin\vartheta_s)} \sum_{m=1}^{N} e^{-j\left(m - \frac{N+1}{2}\right)\pi(\sin\vartheta - \sin\vartheta_s)} \cos\vartheta \, d\vartheta$$

$$= \frac{2\pi}{N^2} \int_{-\pi/2}^{\pi/2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{j(n-m)\pi(\sin\vartheta - \sin\vartheta_s)} \cos\vartheta \, d\vartheta$$

$$= \frac{2\pi}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{-j(n-m)\pi\sin\vartheta_s} \int_{-\pi/2}^{\pi/2} e^{j(n-m)\pi\sin\vartheta} \cos\vartheta \, d\vartheta$$

$$= \frac{2\pi}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{-j(n-m)\pi\sin\vartheta_s} \int_{-1}^{1} e^{j(n-m)\pi u} \, du$$

$$= \frac{2\pi}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{-j(n-m)\pi \sin\vartheta_s} \left. \frac{e^{j(n-m)\pi u}}{j(n-m)\pi} \right|_{-1}^{1}$$

$$= \frac{4\pi}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{-j(n-m)\pi \sin\vartheta_s} \frac{e^{j(n-m)\pi} - e^{-j(n-m)\pi}}{j2(n-m)\pi}$$

$$= \frac{4\pi}{N^2} \sum_{n=1}^{N} \sum_{m=1}^{N} e^{-j(n-m)\pi \sin\vartheta_s} \frac{\sin\left((n-m)\pi\right)}{(n-m)\pi} = \frac{4\pi}{N^2} \sum_{n=1}^{N} 1 = \frac{4\pi}{N}$$

with

$$\mathrm{si}\left((n-m)\pi\right) = \frac{\sin\left((n-m)\pi\right)}{(n-m)\pi} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$$

has been exploited.

5.3 Conventional Beamforming

Now, we consider the case that $x(t,\mathbf{r}_n)$, $n = 1,\ldots,N$ represents the complex envelope of the band-pass signal

$$\tilde{x}(t,\mathbf{r}_n) = \mathrm{Re}\left\{x(t,\mathbf{r}_n)\, e^{j\omega_c t}\right\},\;\; n = 1,\ldots,N$$

where ω_c denotes the carrier frequency.

We assume that the complex envelope $x(t,\mathbf{r}_n)$ is limited to the frequency band $|\omega| \le B/2 = \pi b$.

For a plane wave, that generates the signal $\tilde{s}(t)$ at the origin of the coordinate system, $\tilde{x}(t,\mathbf{r}_n)$ becomes

$$\tilde{x}(t,\mathbf{r}_n) = \tilde{s}(t-\tau_n) = \mathrm{Re}\left\{s(t-\tau_n)\, e^{j\omega_c (t-\tau_n)}\right\},\quad n = 1,\ldots,N.$$

5.3.1 Time Domain Beamforming

Let $T_{\max}$ denote the maximum travel time between any two elements in the array, i.e.

$$\left|\tau_n - \tau_m\right| \leq T_{\max}, \quad \forall n, m = 1, \ldots, N,$$

and let the bandwidth of the complex envelope b be small enough, i.e.

$$bT_{\max} \ll 1,$$

the approximation

$$s(t-\tau_n) \cong s(t)$$

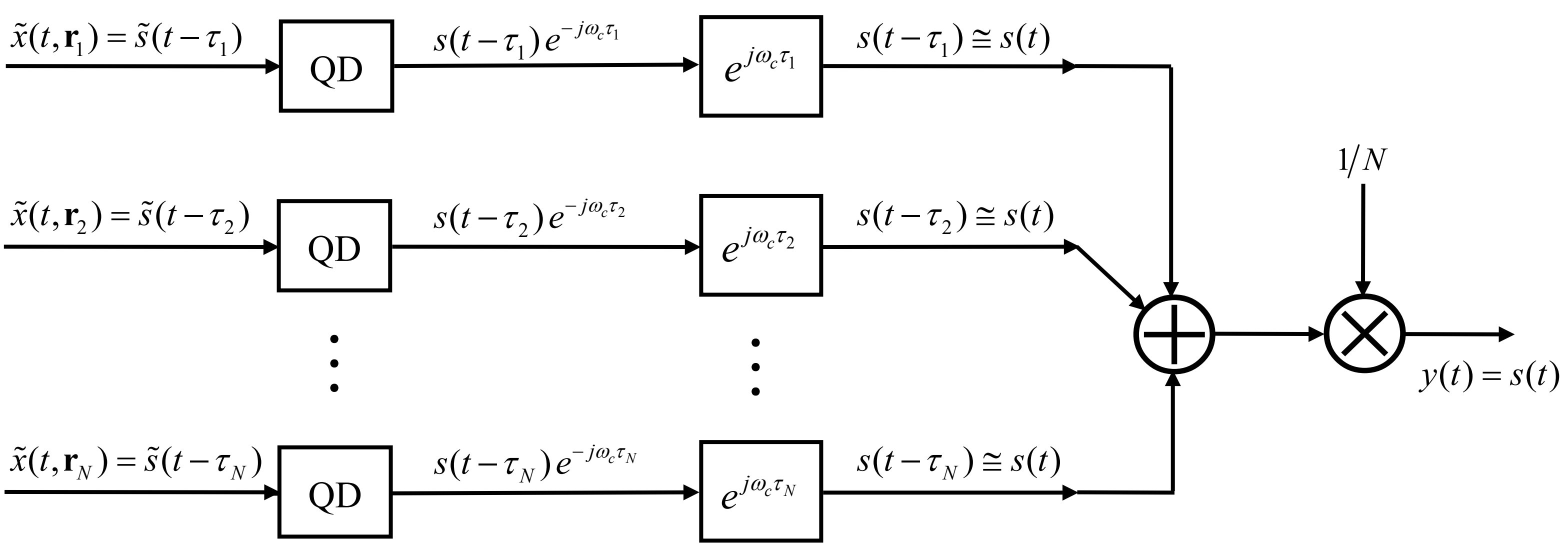
is valid. Hence, we can write

$$\tilde{x}(t, \mathbf{r}_n) = \operatorname{Re}\left\{x(t, \mathbf{r}_n)\, e^{j\omega_c t}\right\} = \operatorname{Re}\left\{s(t-\tau_n)\, e^{j\omega_c (t-\tau_n)}\right\}$$

$$\cong \operatorname{Re}\left\{s(t)\, e^{-j\omega_c \tau_n} e^{j\omega_c t}\right\}, \quad n = 1, \ldots, N.$$

Thus, in the narrow band case, the delays can be approximated by phase shifts, and the conventional beamformer can be implemented by a set of phase shifts instead of delays.

This implementation commonly referred to as phased array beamformer is depicted below.



If $bT_{\max} \ll 1$ is violated one can use delays in conjunction with phase shifters. After the introduction of

$$\tilde{\tau}_n = \tau_n - \tilde{l}_n T_S \quad \text{with} \quad \tilde{l}_n = \left\lfloor \frac{\tau_n}{T_S} + \frac{1}{2} \right\rfloor,$$

we can state that

$$|\tilde{\tau}_n| \leq T_S/2,$$

where T_S denotes in case of

- continuous signals the elementary delay within a tape delay line
- digital signals the sampling period.

Now, assuming

$$bT_{\max} = bT_S \ll 1$$

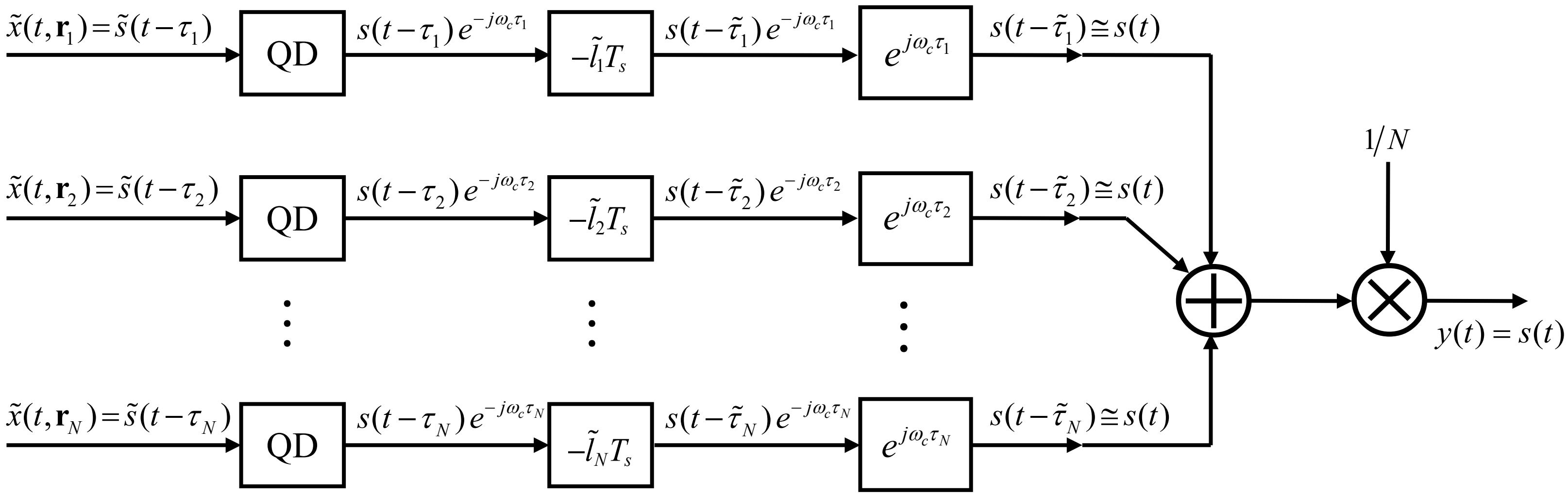
the approximation

$$s(t-\tilde{l}_n T_S-\tilde{\tau}_n)\cong s(t-\tilde{l}_n T_S)$$

holds, and we can write

$$\begin{aligned}\tilde{x}(t,\mathbf{r}_n)&=\mathrm{Re}\left\{x(t,\mathbf{r}_n)\,e^{j\omega_c t}\right\}\\&=\mathrm{Re}\left\{s(t-\tilde{l}_n T_S-\tilde{\tau}_n)\,e^{j\omega_c(t-\tau_n)}\right\}\\&\cong\mathrm{Re}\left\{s(t-\tilde{l}_n T_S)\,e^{-j\omega_c\tau_n}\,e^{j\omega_c t}\right\},\quad n=1,\ldots,N.\end{aligned}$$

The combined use of delays and phase shifters, as visualized in the following figure, represents a beamformer implementation that is typically employed in sonar applications.



However, if f_S is close to the Nyquist rate, e.g. complex sampling with $f_S = 3/2\,b$ and therefore

$$bT_S = 2f_S/3\cdot T_S = 2/3 \not\ll 1,$$

more sophisticated interpolation techniques are required for reconstructing the signal $s(t)$ in each channel, satisfactorily.

Example:

We suppose an array of $n = 1,\ldots, N$ receivers, where

$$\tilde{x}_n(t) = \mathrm{Re}\left\{ x_n(t)\, e^{j\omega_c t} \right\} \text{ with } x_n(t) = x_{I,n}(t) + j x_{Q,n}(t)$$

denotes the signal measured with the n-th receiver and where

$$\tilde{s}(t) = \mathrm{Re}\left\{ s(t)\, e^{j\omega_c t} \right\} \text{ with } s(t) = s_I(t) + j s_Q(t)$$

represents the signal of interest/target which for

a) passive sonar and far-field active sonar applications is supposed to be measured at a reference location, e.g. at the origin of the coordinate system,

b) near-field active sonar applications is given by the transmitted signal which has been initiated at $t = 0$.

In a noiseless case the signal at the n-th receiver is given by

$$\tilde{x}_n(t) = \tilde{s}(t-\tau_n) = \mathrm{Re}\left\{ s(t-\tau_n)\, e^{j\omega_c (t-\tau_n)} \right\} \;\Rightarrow\; x_n(t) = s(t-\tau_n)\, e^{-j\omega_c \tau_n}.$$

Hence, a reconstruction of $s(t)$ using $x_n(t)$ is provided by

$$s(t) = x_n(t+\tau_n)\, e^{j\omega_c \tau_n} = \left(x_{I,n}(t+\tau_n) + j x_{Q,n}(t+\tau_n) \right) e^{j\omega_c \tau_n}.$$

Considering $s(t)$ at time instances $t = lT_S$, we can write

$$\begin{aligned} s(lT_S) &= x_n(lT_S+\tau_n)\, e^{j\omega_c \tau_n} = x_n(l_n T_S + \tilde{\tau}_n)\, e^{j\omega_c \tau_n} \\ &= \left(x_{I,n}(l_n T_S + \tilde{\tau}_n) + j x_{Q,n}(l_n T_S + \tilde{\tau}_n) \right) e^{j\omega_c \tau_n}, \end{aligned}$$

where

$$l_n = l + \tilde{l}_n \quad \text{and} \quad \tilde{\tau}_n = \tau_n - \tilde{l}_n T_S$$

with $\tilde{l}_n T_S \le \tau_n < (\tilde{l}_n + 1) T_S$ and $\tilde{l}_n \in \mathbb{Z}$.

Since for $x_n(t)$ only the sample values $x_n(l_n T_S)$ are available the

$$x_n(l_n T_S + \tilde{\tau}_n) = x_{I,n}(l_n T_S + \tilde{\tau}_n) + j x_{Q,n}(l_n T_S + \tilde{\tau}_n)$$

have to be approximately determined by interpolation. After restricting our self to the linear interpolation approach

$$x_{I,n}(l_n T_S + \tilde{\tau}_n) \cong (1-\alpha_n) x_{I,n}(l_n T_S) + \alpha_n \, x_{I,n}\big((l_n + 1) T_S\big)$$

$$x_{Q,n}(l_n T_S + \tilde{\tau}_n) \cong (1-\alpha_n) x_{Q,n}(l_n T_S) + \alpha_n \, x_{Q,n}\big((l_n + 1) T_S\big),$$

where $\alpha_n = \tilde{\tau}_n / T_S$, the beamformer output signal can be expressed by

$$y(lT_S) = \frac{1}{N} \sum_{n=1}^{N} \Big\{ \Big[(1-\alpha_n) x_{I,n}(l_n T_S) + \alpha_n \, x_{I,n}\big((l_n+1)T_S\big) \Big] + \\ + j \Big[(1-\alpha_n) x_{Q,n}(l_n T_S) + \alpha_n \, x_{Q,n}\big((l_n+1)T_S\big) \Big] \Big\} e^{j\omega_c \tau_n}.$$

5.3.2 Frequency Domain Beamformer

Fast convolution

Let x_l and h_l $(l = 0,\ldots,L-1)$ be a finite input sequence and a finite discrete time impulse response of a filter, respectively.

Exploiting that the discrete Fourier transform (DFT) of a circular convolution can be expressed by

$$(Y_k) = \mathrm{DFT}\{(y_l)\} = \mathrm{DFT}\left\{\left(\sum_{i=0}^{L-1}[x_{l-i}]_L h_i\right)\right\}$$

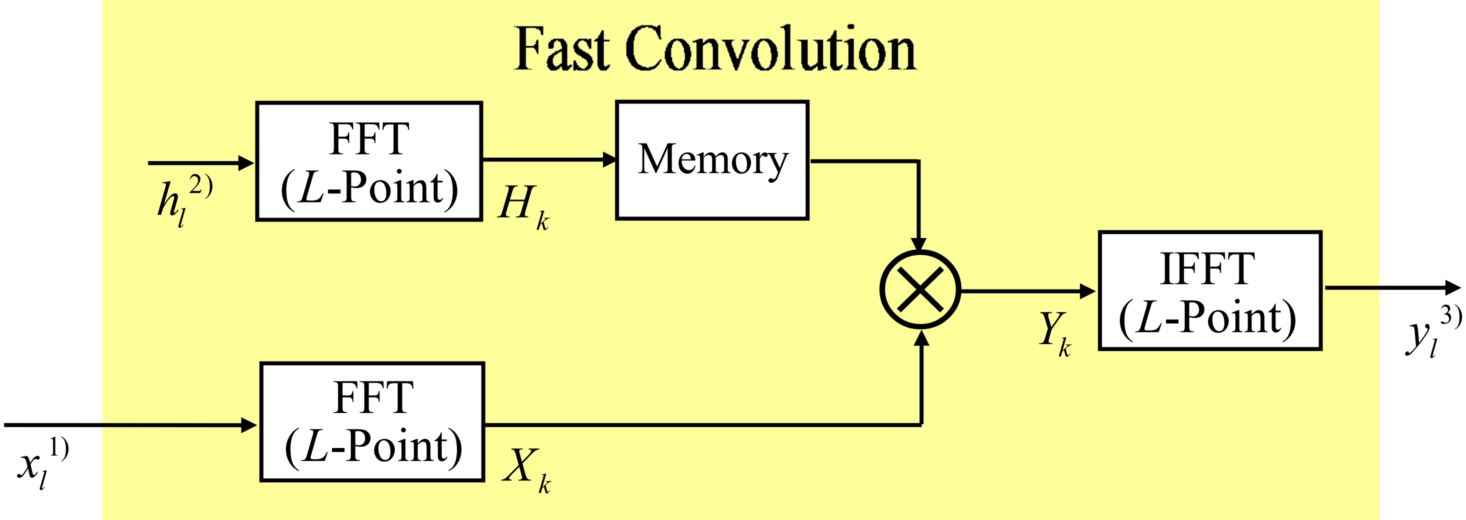
$$= \mathrm{DFT}\left\{\left(\sum_{i=0}^{L-1} x_i [h_{l-i}]_L\right)\right\}$$

$$= \mathrm{DFT}\{(x_i)\}\,\mathrm{DFT}\{(h_i)\} = (X_k H_k),$$

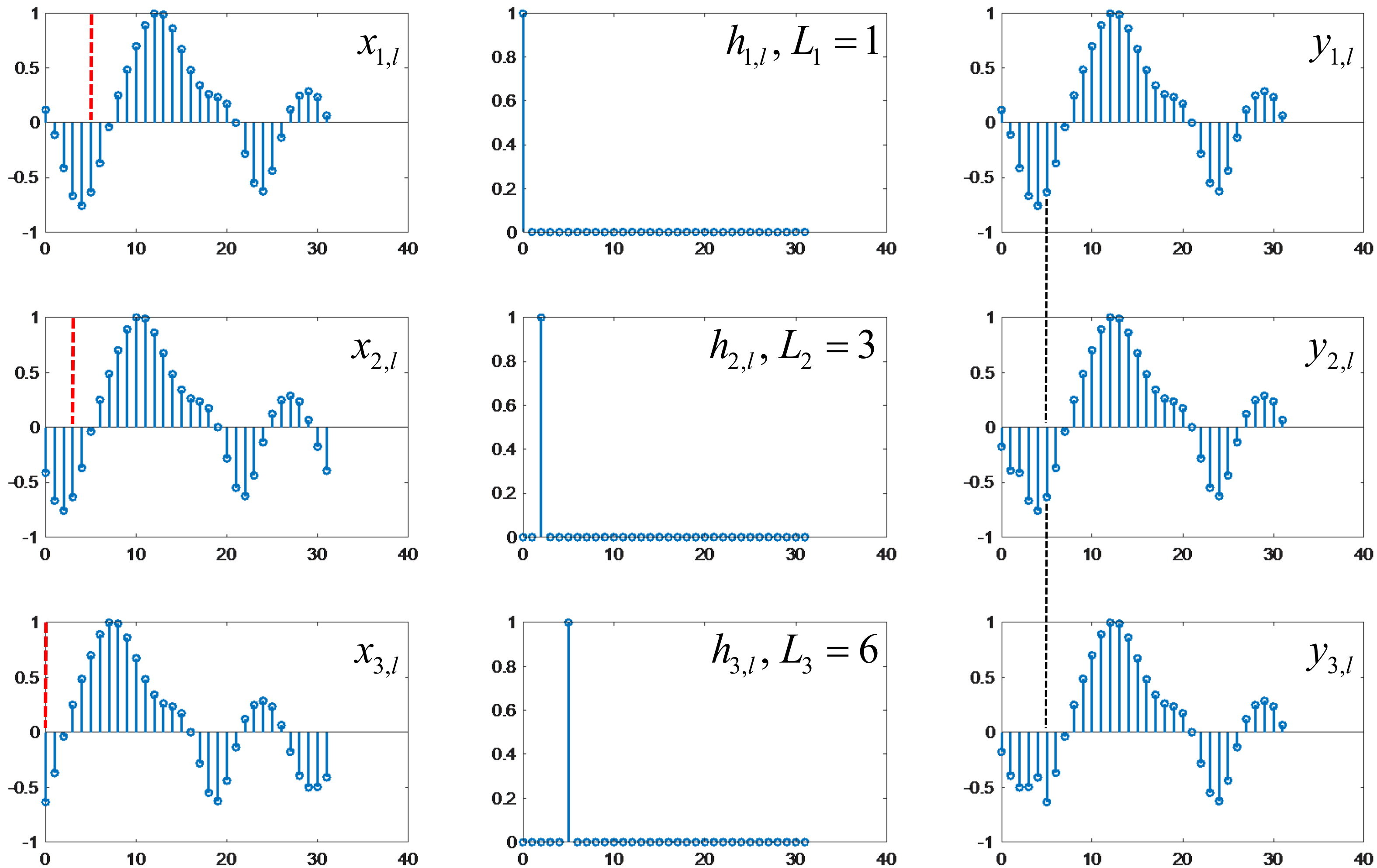
where

$$l,k=0,\dots,L-1 \text{ and } \tilde{u}_j=\left[u_j\right]_L \text{ with } \tilde{u}_j=\tilde{u}_{j+L},\ \forall j\in\mathbb{Z},$$

the linear convolution can be efficiently carried out via the fast Fourier transform (FFT) after suitable truncation/zero padding.

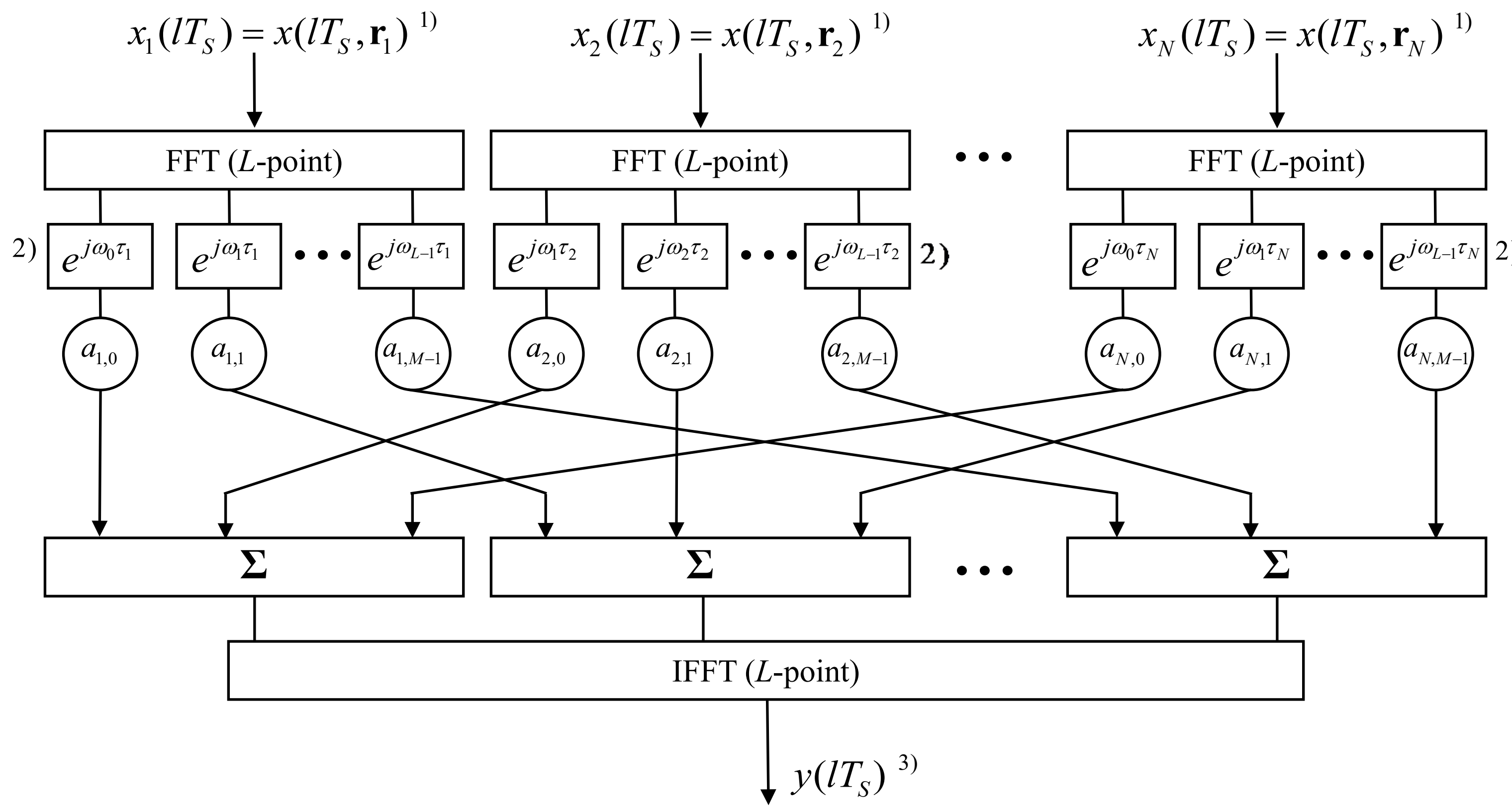


1) $l=0,\dots,L-1$, 2) $l=0,\dots,\tilde{L}-1$, 3) $l=\tilde{L}-1,\dots,L-1$



$L = 32$ $\qquad \tilde{L}_{\max} = \max\{L_1, L_2, L_3\} = 6$ $\qquad l = \tilde{L}_{\max} - 1, \ldots, L - 1$

FFT-Beamformer



[1] $l = 0,\ldots,L-1$, [2] $\omega_k = \omega_S k/L + \omega_c$ for $k = 0,\ldots,L-1$, [3] $l = \left\lceil \max\limits_{1\le n\le N}(\tau_n)/T_S \right\rceil,\ldots,L-1$,

The following items often prevent a direct application of the FFT based convolution.

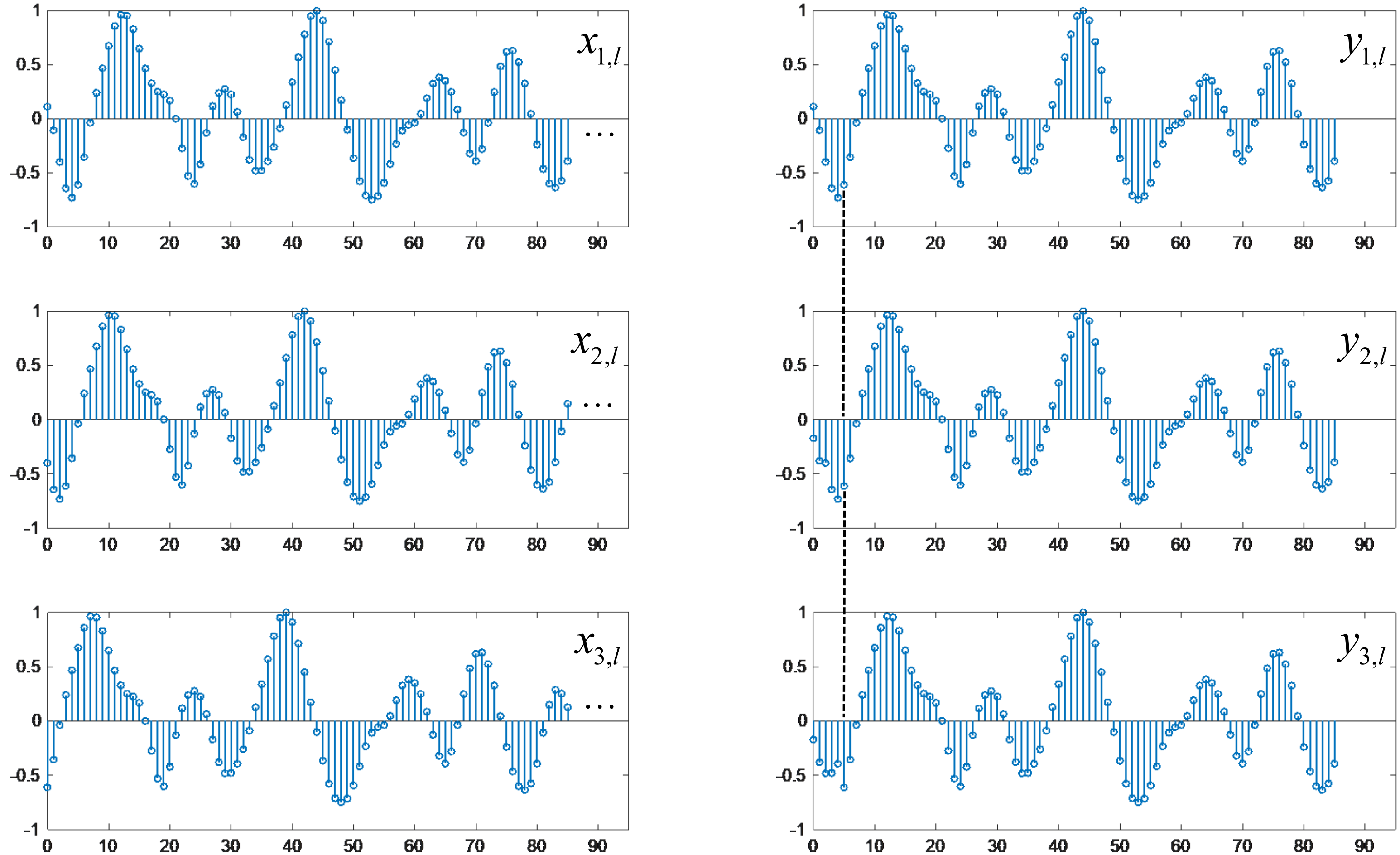
- Long Data sets, e.g. if long ping periods (active sonar) or long integration times (passive sonar) are of interest
- Motion Compensation during Beamforming, e.g. new motion data are typically every 20 ms available
- Dynamic wave front curvature compensation, e.g. required for imaging sonars in near-field applications

Consequently, a linear convolution scheme based on sequen-ces of Fourier transformed data sets is required.

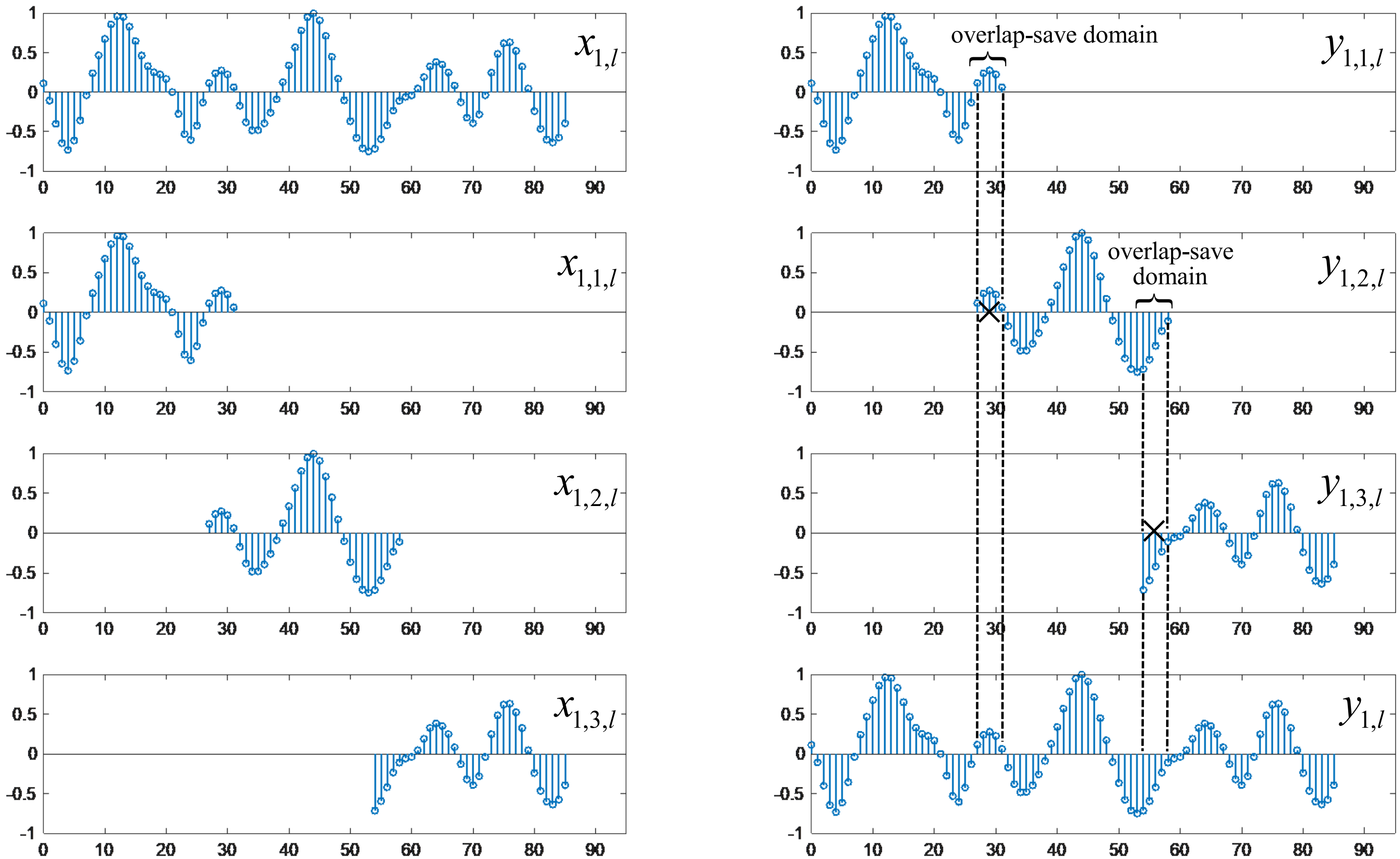
A suitable approach is provided by exploiting the overlap-save or the overlap-add method as visualized below.

Overlap-Save Method

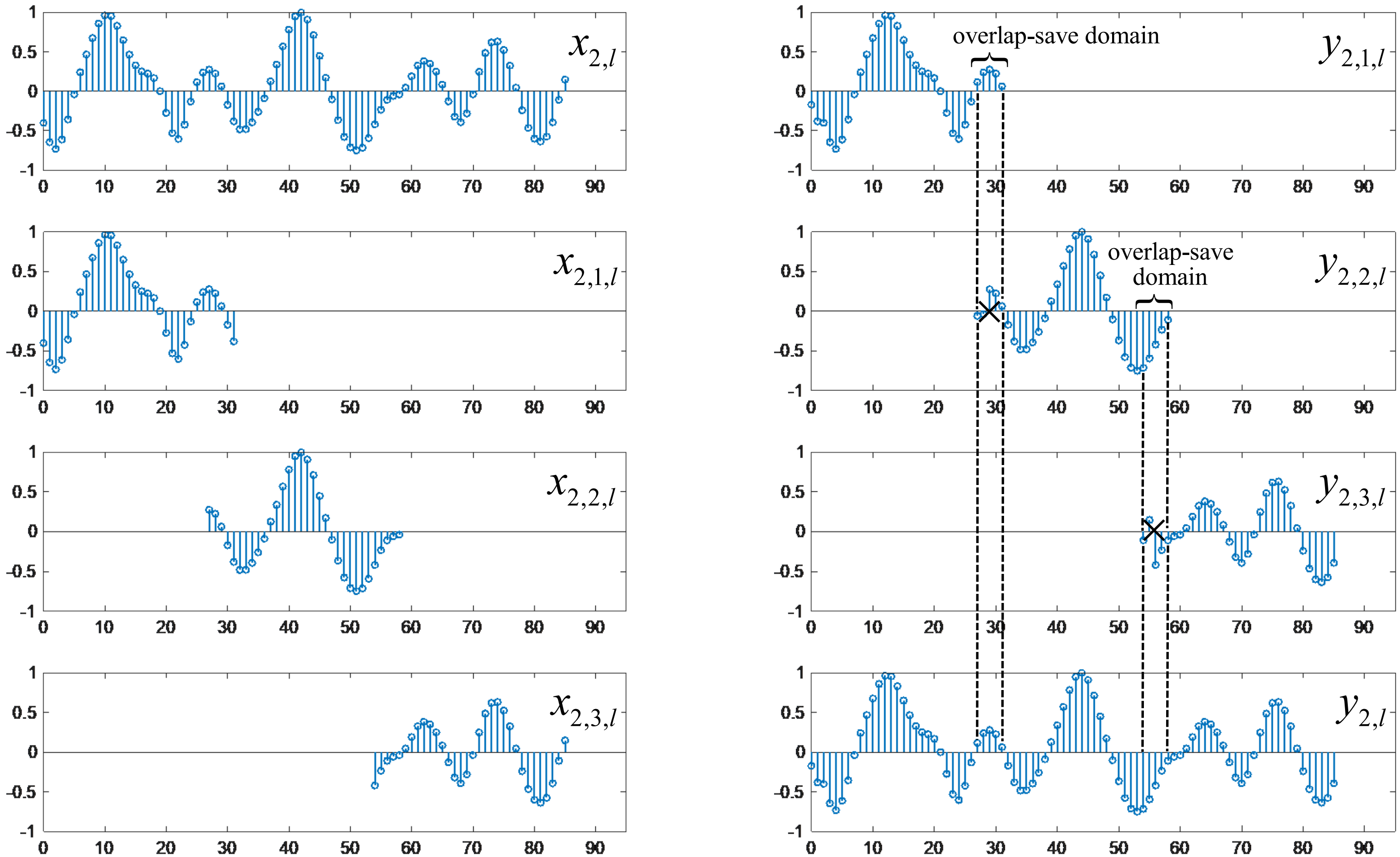
Input Signals | Aligned Output Signals



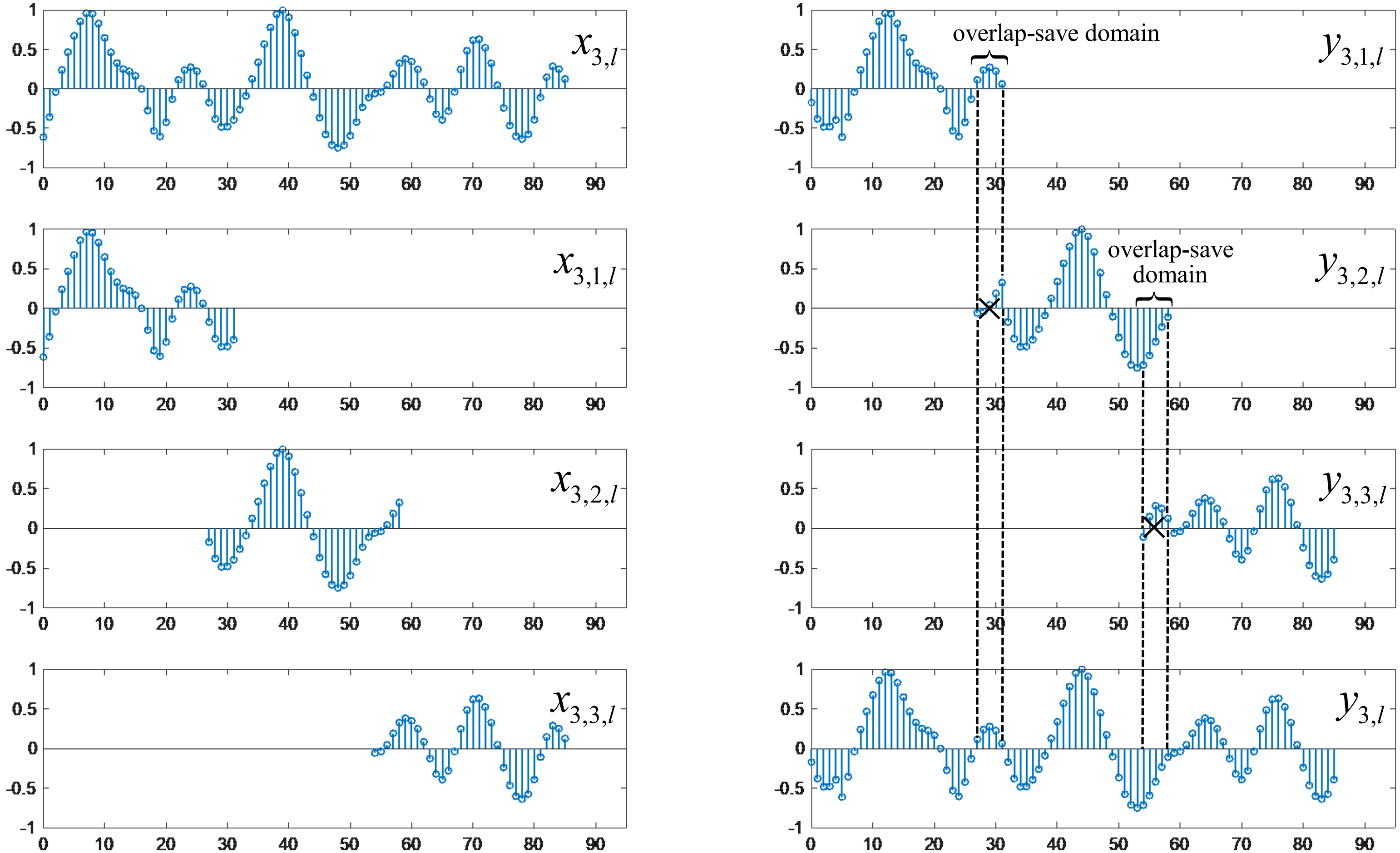
Overlap-Save Method / Processing of $x_{1,l}$ to get $y_{1,l}$



Overlap-Save Method / Processing of $x_{2,l}$ to get $y_{2,l}$

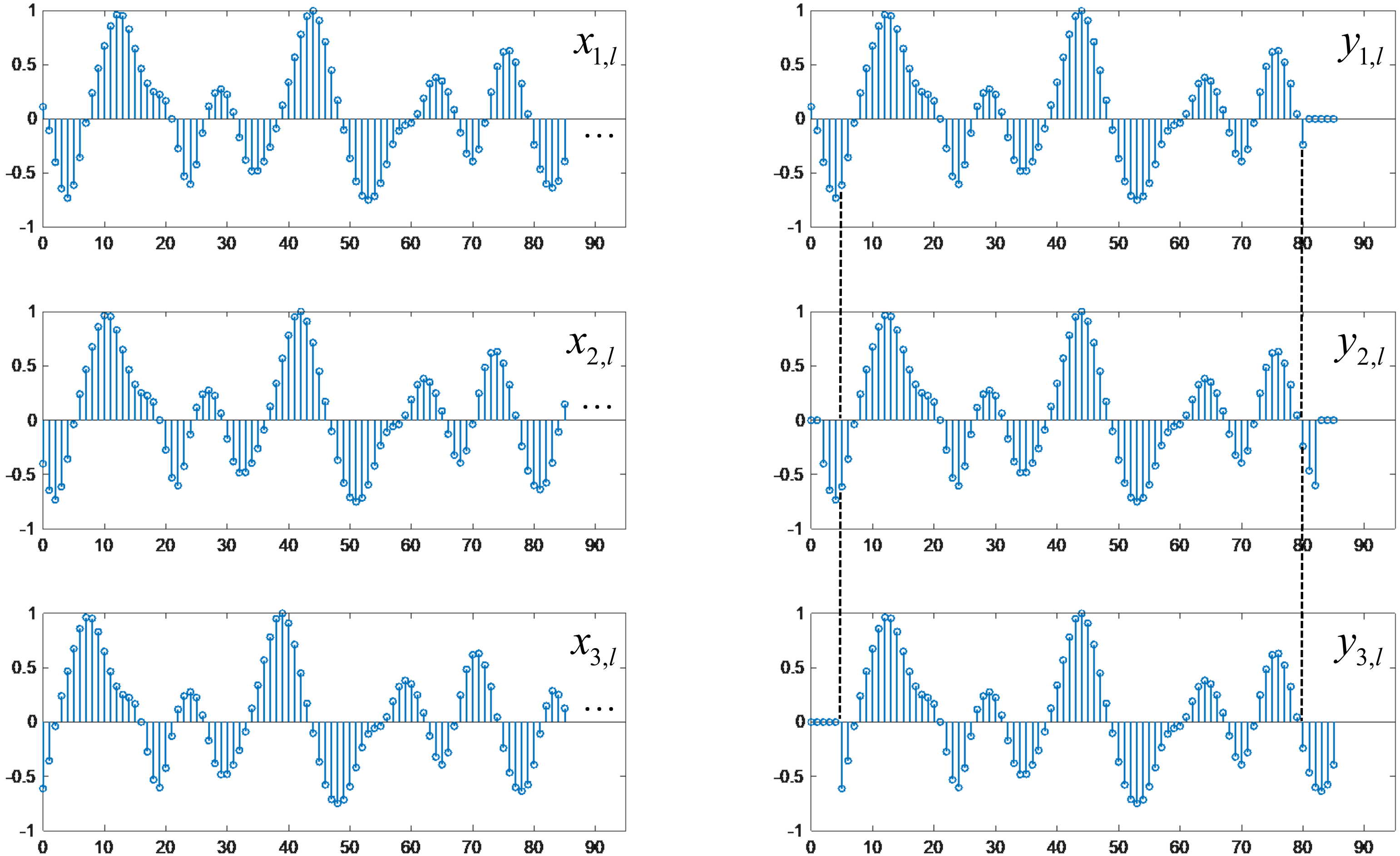


Overlap-Save Method / Processing of $x_{3,l}$ to get $y_{3,l}$

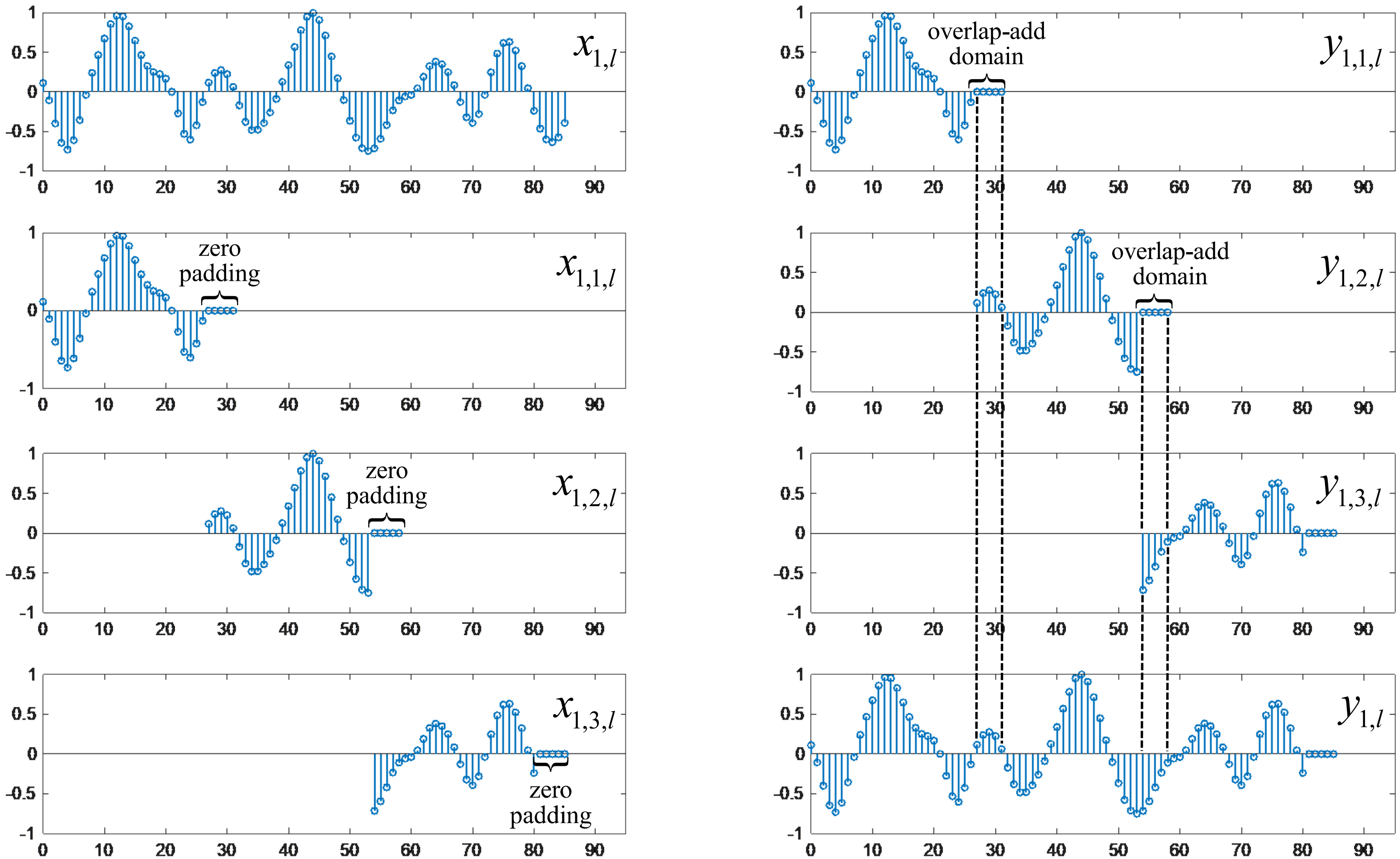


Overlap-Add Method

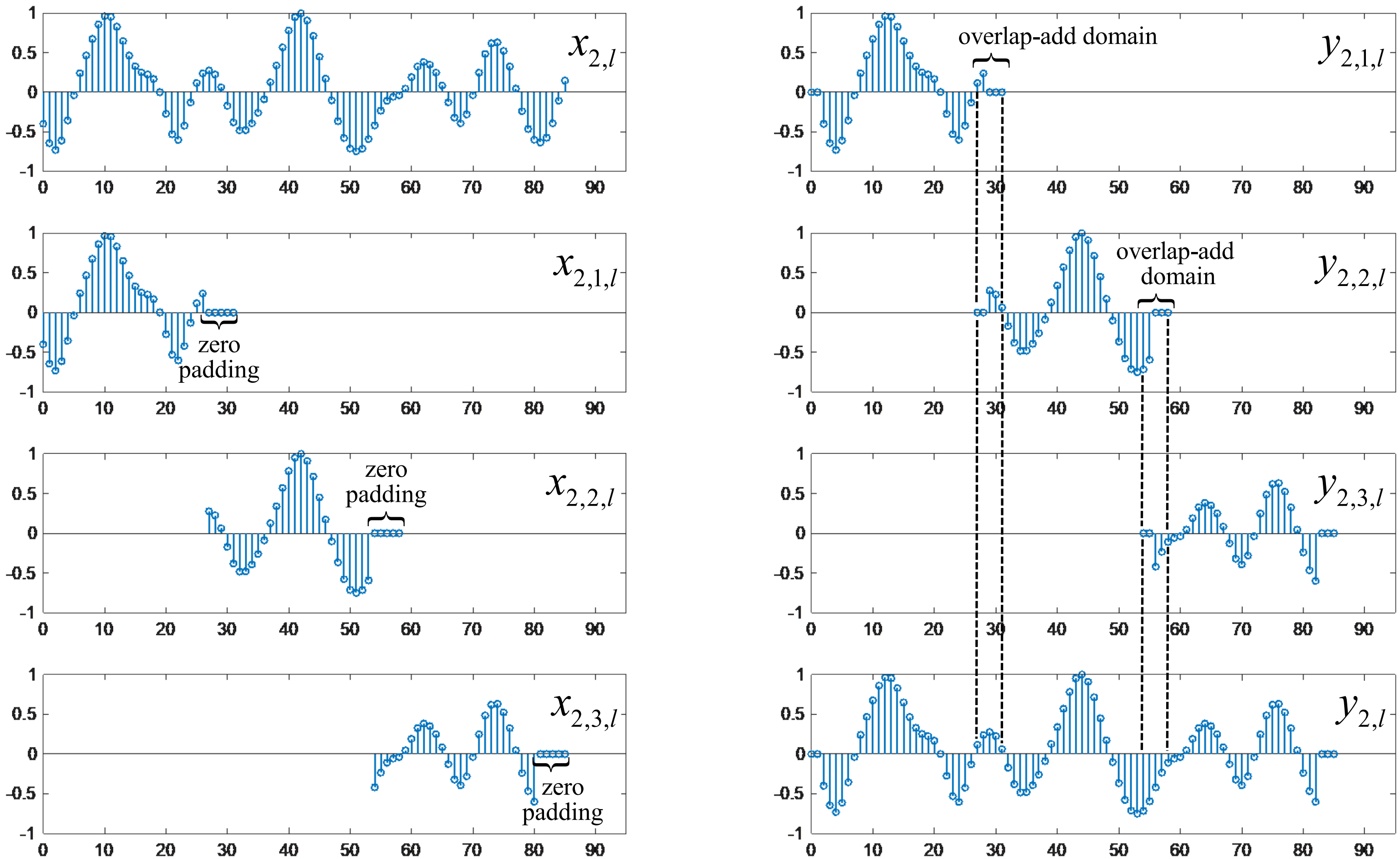
Input Signals | Aligned Output Signals



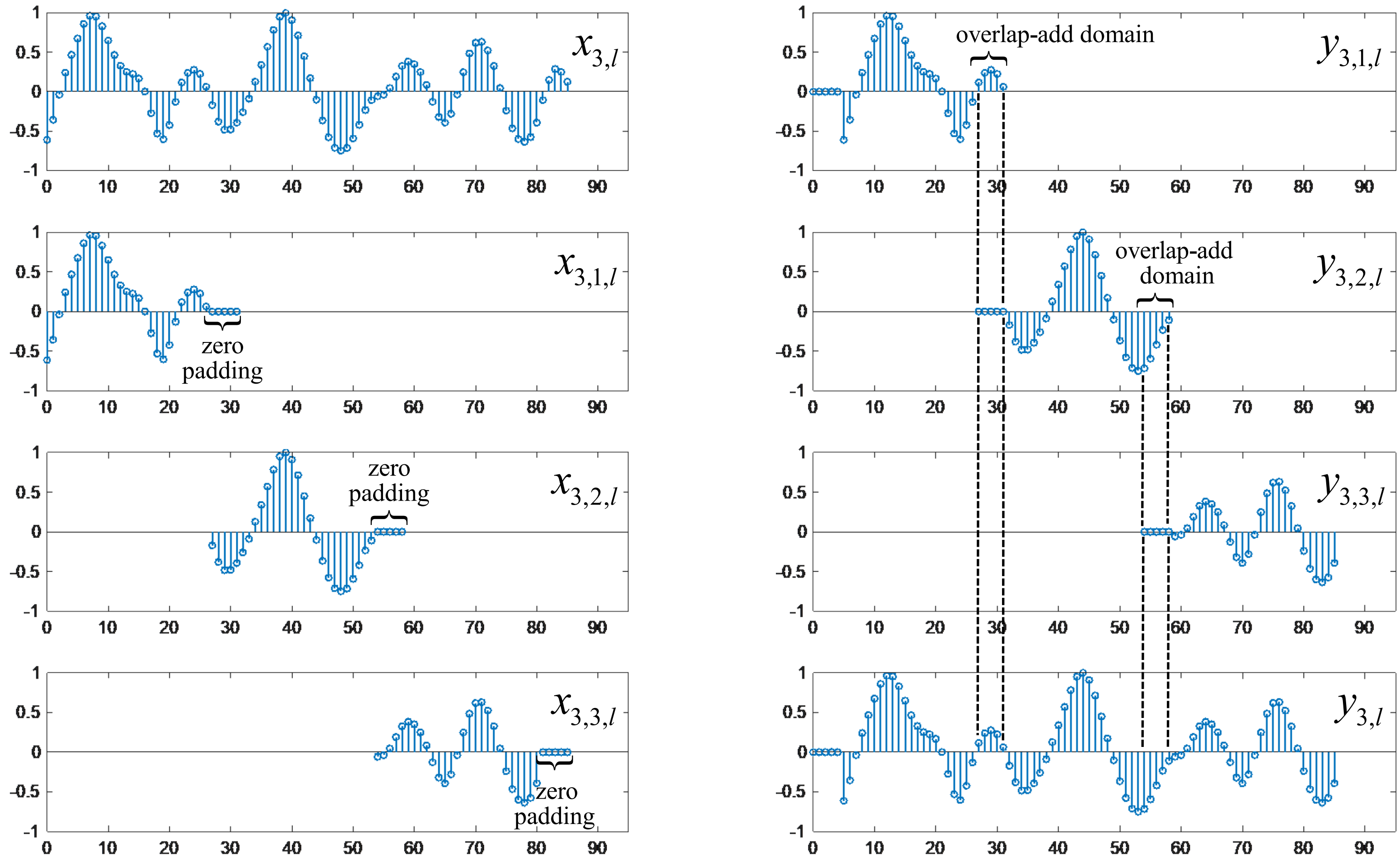
Overlap-Add Method / Processing of $x_{1,l}$ to get $y_{1,l}$



Overlap-Add Method / Processing of $x_{2,l}$ to get $y_{2,l}$



Overlap-Add Method / Processing of $x_{3,l}$ to get $y_{3,l}$



5.4 Introduction to high resolution methods

5.4.1 Narrowband snapshot model

In the following a time-domain model appropriate for narrow-band waveforms is developed. First, a wave field generated by a single plane wave is considered. Spatial sampling of this wave field by an array of sensors provides the complex signals

$$x(t,\mathbf{r}_n) = s(t-\tau_n)\, e^{j\omega(t-\tau_n)},\;\; \tau_n = -\mathbf{k}_s^T \mathbf{r}_n / \omega,\;\; n = 1,\ldots,N.$$

Since the wave field is supposed to be narrowband, i.e.

$$s(t) \cong s(t-\tau_n),\;\; \forall n = 1,\ldots,N,$$

we can approximately write

$$x(t,\mathbf{r}_n) \cong s(t)\, e^{j(\omega t + \mathbf{k}_s^T \mathbf{r}_n)},\;\; n = 1,\ldots,N$$

and in vector notation

$$\mathbf{x}(t) = \left(x(t,\mathbf{r}_1),\ldots,x(t,\mathbf{r}_N)\right)^T = s(t)\,e^{j\omega t}\,\mathbf{a}(\mathbf{k}_s),$$

where

$$\mathbf{a}(\mathbf{k}_s) = \left(\exp(j\mathbf{k}_s^T\mathbf{r}_1),\ldots,\exp(j\mathbf{k}_s^T\mathbf{r}_N)\right)^T.$$

Let $\mathbf{x}(t)$ be a zero-mean vector valued stochastic process, i.e.

$$\mathrm{E}\left(\mathbf{x}(t)\right) = \mathbf{0}$$

then, its matrix valued covariance function defined by

$$\mathbf{c}_{\mathbf{xx}}(\tau) = \mathrm{E}\left(\mathbf{x}(t)\mathbf{x}^H(t-\tau)\right)$$

provides for $\tau = 0$

$$\mathbf{c}_{\mathbf{xx}} = \mathbf{c}_{\mathbf{xx}}(0) = \mathrm{E}\left|s(t)\right|^2\mathbf{a}(\mathbf{k}_s)\mathbf{a}^H(\mathbf{k}_s) = \sigma_s^2\,\mathbf{a}(\mathbf{k}_s)\mathbf{a}^H(\mathbf{k}_s).$$

Now, M narrowband sources are assumed. Each source is emitting a plane-wave with envelope $s_m(t)$ and wave vector $\mathbf{k}_m$ for $m = 1,\ldots,M$.

The resulting wave field is measured by an array of sensors, where the vector of the senor signals can be expressed by

$$\mathbf{x}(t) = \sum_{m=1}^{M} s_m(t)\,\mathbf{a}(\mathbf{k}_m)\,e^{j\omega t} = \left(\mathbf{a}(\mathbf{k}_1),\ldots,\mathbf{a}(\mathbf{k}_M)\right)\begin{pmatrix} s_1(t) \\ \vdots \\ s_M(t) \end{pmatrix} e^{j\omega t}$$

$$= \mathbf{A}\,\mathbf{s}(t)\,e^{j\omega t},$$

with

$$\mathbf{A} = \left(\mathbf{a}(\mathbf{k}_1),\ldots,\mathbf{a}(\mathbf{k}_M)\right) \text{ and } \mathbf{s}(t) = \left(s_1(t),\ldots,s_M(t)\right)^T.$$

Thus, the covariance matrix of the signal vector $\mathbf{x}(t)$ becomes

$$\mathbf{c}_{\mathbf{xx}} = \mathrm{E}\left(\mathbf{x}(t)\,\mathbf{x}^H(t)\right) = \mathrm{E}\left(\mathbf{A}\mathbf{s}(t)\left(\mathbf{A}\mathbf{s}(t)\right)^H\right)$$
$$= \mathrm{E}\left(\mathbf{A}\mathbf{s}(t)\,\mathbf{s}^H(t)\mathbf{A}^H\right) = \mathbf{A}\,\mathrm{E}\left(\mathbf{s}(t)\,\mathbf{s}^H(t)\right)\mathbf{A}^H = \mathbf{A}\,\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H.$$

If the M sources are supposed to be uncorrelated, i.e.

$$\mathbf{c}_{\mathbf{ss}} = \mathrm{E}\left(\mathbf{s}(t)\,\mathbf{s}(t)^H\right) = \mathrm{diag}\left(\sigma_{s_1}^2,\ldots,\sigma_{s_M}^2\right)$$

since

$$\mathrm{E}\left|s_m(t)\right|^2 = \sigma_{s_m}^2 \quad \text{and} \quad \mathrm{E}\left(s_m(t)\,s_n^H(t)\right) = 0, \;\; m \neq n,$$

we can write

$$\mathbf{c}_{\mathbf{xx}} = \mathbf{A}\,\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H = \sum_{m=1}^{M} \sigma_{s_m}^2\,\mathbf{a}(\mathbf{k}_m)\,\mathbf{a}^H(\mathbf{k}_m).$$

Generalization of the pervious results to the case of M uncorrelated sources imbedded in uncorrelated additive noise gives

$$\mathbf{x}(t) = \left(\sum_{m=1}^{M} s_m(t)\,\mathbf{a}(\mathbf{k}_m) + \mathbf{u}(t) \right) e^{j\omega t} = \left(\mathbf{A}\,\mathbf{s}(t) + \mathbf{u}(t) \right) e^{j\omega t}$$

for the signal vector and

$$\mathbf{c}_{\mathbf{xx}} = \mathbf{A}\,\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H + \mathbf{c}_{\mathbf{uu}} = \sum_{m=1}^{M} \sigma_{s_m}^2\,\mathbf{a}(\mathbf{k}_m)\,\mathbf{a}^H(\mathbf{k}_m) + \mathbf{c}_{\mathbf{uu}}$$

with $\mathrm{E}\left(\mathbf{u}(t)\right) = \mathbf{0}$ and $\mathbf{c}_{\mathbf{uu}} = \mathrm{E}\left(\mathbf{u}(t)\,\mathbf{u}^H(t)\right)$

for the corresponding covariance matrix.

Furthermore, supposing the noise to be spatially white, e.g. if isotropic ambient noise is impinging on a $\lambda/2$ equidistantly spaced line array, we obtain

$$\mathbf{c}_{\mathbf{xx}} = \mathbf{A}\,\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H + \sigma_{\mathbf{u}}^2\mathbf{I} = \sum_{m=1}^{M} \sigma_{s_m}^2 \mathbf{a}(\mathbf{k}_m)\,\mathbf{a}^H(\mathbf{k}_m) + \sigma_{\mathbf{u}}^2\mathbf{I},$$

where $\sigma_{\mathbf{u}}^2$ denotes the noise power (variance) in the frequency band of interest $[\omega - B/2, \omega + B/2]$ for each channel.

Moreover, if the components of $\mathbf{s}(t)$ and $\mathbf{u}(t)$ have flat spectra, i.e. are nearly white over B, then

$$\mathbf{c}_{\mathbf{xx}}(lT_S) = \mathrm{E}\left(\mathbf{x}(t)\,\mathbf{x}^H(t - lT_S)\right) \cong \mathbf{0} \quad \text{for} \quad T_S = 1/b \quad \text{and} \quad l \neq 0.$$

Thus, $\mathbf{x}(lT_S)$, $l = 0, 1, \ldots$ represents a sequence of uncorrelated, zero-mean complex random vectors whose covariance matrix is given by $\mathbf{c}_{\mathbf{xx}}$ as expressed above.

Finally, if $\mathbf{x}(lT_S)$ satisfies the aforementioned properties, the covariance matrix $\mathbf{c_{xx}}$ can be consistently estimated by

$$\hat{\mathbf{c}}_{\mathbf{xx}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{x}(lT_S)\,\mathbf{x}^H(lT_S).$$

5.4.2 Classical Beamformer

The classical beamformer is defined by the criterion

$$\begin{aligned} q_{CB}(\mathbf{k}) &= \mathbf{a}^H(\mathbf{k})\,\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{a}(\mathbf{k}) \\ &= \sum_{p=1}^{N}\sum_{q=1}^{N} a_p^*(\mathbf{k})\,a_q(\mathbf{k})\hat{c}_{pq} \end{aligned}$$

with $\mathbf{a}(\mathbf{k}) = \left(a_1(\mathbf{k}),\ldots,a_N(\mathbf{k})\right)^T$ and $\hat{\mathbf{c}}_{\mathbf{xx}} = \left(\hat{c}_{pq}\right)_{p,q=1,\ldots,N}$.

The following reformulations show, that the classical beamformer criterion can be interpreted as an non-parametric estimate of the wave number spectrum $C_{\mathbf{xx}}(\mathbf{k})$ of the wave field $x(t,\mathbf{r})$.

$$q_{CB}(\mathbf{k}) = \mathbf{a}^H(\mathbf{k})\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{a}(\mathbf{k}) = \sum_{p=1}^{N}\sum_{q=1}^{N} a_p^*(\mathbf{k})\, a_q(\mathbf{k})\, \hat{c}_{pq}$$

$$= \sum_{p=1}^{N}\sum_{q=1}^{N}\left(e^{-j\mathbf{k}^T\mathbf{r}_p} e^{j\mathbf{k}^T\mathbf{r}_q} \frac{1}{L}\sum_{l=0}^{L-1} x_p(lT_S)\, x_q^*(lT_S) \right)$$

$$= \frac{1}{L}\sum_{l=0}^{L-1}\left(\sum_{p=1}^{N} x(lT_S,\mathbf{r}_p)\, e^{-j\mathbf{k}^T\mathbf{r}_p} \sum_{q=1}^{N} x^*(lT_S,\mathbf{r}_q)\, e^{j\mathbf{k}^T\mathbf{r}_q} \right)$$

$$= \frac{1}{L}\sum_{l=0}^{L-1} X(\mathbf{k},l)\, X^*(\mathbf{k},l) = \frac{1}{L}\sum_{l=0}^{L-1} \left| X(\mathbf{k},l) \right|^2,$$

where

$$a_n(\mathbf{k}) = e^{j\mathbf{k}^T\mathbf{r}_n} \quad \text{and} \quad X(\mathbf{k},l) = \sum_{n=1}^{N} x(lT_S,\mathbf{r}_n)\, e^{-j\mathbf{k}^T\mathbf{r}_n} = \sum_{n=1}^{N} x_n(lT_S)\, e^{-j\mathbf{k}^T\mathbf{r}_n}$$

has been exploited. As $X(\mathbf{k},l)$ represents the spatial-discrete Fourier transform of each snapshot,

$$I_{\mathbf{xx}}(\mathbf{k},l) = \frac{1}{N}\left|X(\mathbf{k},l)\right|^2, \;\; l = 1,\ldots,L$$

can be considered as the corresponding periodogram in the wave number domain.

Hence, averaging over periodograms of consecutive snapshots provides the wave number spectrum estimate

$$q_{CB}(\mathbf{k}) = \mathbf{a}^H(\mathbf{k})\,\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{a}(\mathbf{k}) = \frac{1}{L}\sum_{l=0}^{L-1}\left|X(\mathbf{k},l)\right|^2 = \frac{N}{L}\sum_{l=0}^{L-1} I_{\mathbf{xx}}(\mathbf{k},l) = N\hat{C}_{\mathbf{xx}}(\mathbf{k}).$$

For a given $k = \omega/c$ the classical beamformer criterion can be expressed as a function of azimuth φ and elevation ϑ by

$$\begin{aligned}\tilde{q}_{CB}(\varphi,\vartheta) &= q_{CB}\left(k\boldsymbol{\xi}(\varphi,\vartheta)\right) = \mathbf{a}^H\left(k\boldsymbol{\xi}(\varphi,\vartheta)\right)\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{a}\left(k\boldsymbol{\xi}(\varphi,\vartheta)\right) \\ &= \tilde{\mathbf{a}}^H(\varphi,\vartheta)\hat{\mathbf{c}}_{\mathbf{xx}}\tilde{\mathbf{a}}(\varphi,\vartheta)\end{aligned}$$

with

$$\begin{aligned}\tilde{\mathbf{a}}(\varphi,\vartheta) &= \mathbf{a}\left(k\boldsymbol{\xi}(\varphi,\vartheta)\right) = \left(a_1\left(k\boldsymbol{\xi}(\varphi,\vartheta)\right),\ldots,a_N\left(k\boldsymbol{\xi}(\varphi,\vartheta)\right)\right)^T \\ &= \left(\exp\left(jk\boldsymbol{\xi}^T(\varphi,\vartheta)\mathbf{r}_1\right),\ldots,\exp\left(jk\boldsymbol{\xi}^T(\varphi,\vartheta)\mathbf{r}_N\right)\right)^T.\end{aligned}$$

Finally, the directions of arrivals (DOAs) of the plan waves of interest are indicated by the locations of the peaks in the classical beamformer diagram, i.e. in the versus azimuth φ and elevation ϑ visualized classical beamformer criterion.

Estimate updating: *(covariance matrix)*

Growing window

$$\hat{\mathbf{c}}_{\mathbf{xx},l}^{G} = \frac{1}{l+1}\left(l\,\hat{\mathbf{c}}_{\mathbf{xx},l-1}^{G} + \mathbf{x}(lT_s)\,\mathbf{x}^H(lT_s)\right) \text{ with } \hat{\mathbf{c}}_{\mathbf{xx},0}^{G} = \mathbf{x}(0)\,\mathbf{x}^H(0)$$

Sliding window of length L

$$\hat{\mathbf{c}}_{\mathbf{xx},l}^{S} = \hat{\mathbf{c}}_{\mathbf{xx},l}^{G} \quad \text{for } \; l < L \; \text{ else}$$

$$\hat{\mathbf{c}}_{\mathbf{xx},l}^{S} = \frac{1}{L}\left(L\,\hat{\mathbf{c}}_{\mathbf{xx},l-1}^{S} - \mathbf{x}((l-L)T_s)\,\mathbf{x}^H((l-L)T_s) + \mathbf{x}(lT_s)\,\mathbf{x}^H(lT_s)\right)$$

Exponential smoothing

$$\hat{\mathbf{c}}_{\mathbf{xx},l}^{E} = (1-\beta)\,\hat{\mathbf{c}}_{\mathbf{xx},l-1}^{E} + \beta\,\mathbf{x}(lT_s)\,\mathbf{x}^H(lT_s)$$

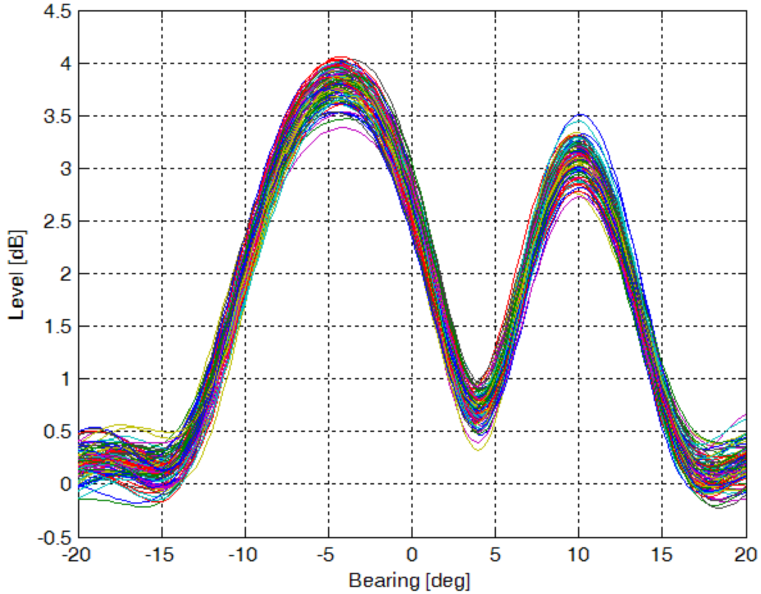
$$\text{with } \; \hat{\mathbf{c}}_{\mathbf{xx},0}^{E} = \mathbf{x}(0)\,\mathbf{x}^H(0) \; \text{ and } \; 0 < \beta < 1$$

Simulation Results of Classical Beamforming

$\varphi_1 = -7°,\ \varphi_2 = -2°,\ \varphi_3 = 10°$

$L = 1000,\ \ N = 15$

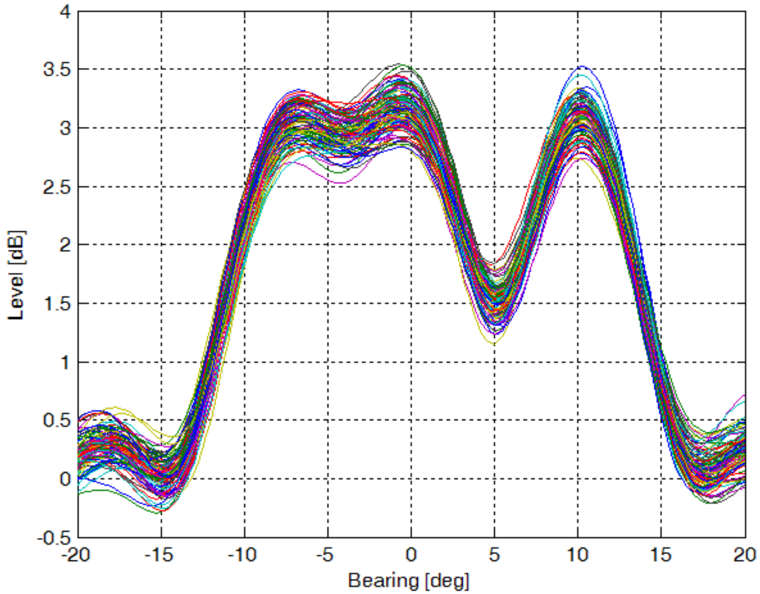
$SNR = -10\log_{10}(N)\,\text{dB}$



$\varphi_1 = -7°,\ \varphi_2 = 0°,\ \varphi_3 = 10°$

$L = 1000,\ \ N = 15$

$SNR = -10\log_{10}(N)\,\text{dB}$



Remarks: Classical Beamforming

Assets

- Number of sources has not to be known
- Computationally fairly simple
- Robust against model mismatches

Drawbacks

- Rather poor resolving power
- Narrowband / generalization to broadband not straight forward
- Estimates signal + noise power only

The DOA resolving power of the classical beamformer is rather limited. Therefore, a variety of advanced methods have been developed for enhancing the resolution capabilities.

In the following, we briefly focus on methods that produce diagrams that are comparable to those of the classical beamformer, however, that should have sharper peaks at points indicating sources. Any such diagram computed from an estimate of the covariance matrix $\mathbf{c}_{\mathbf{xx}}$ is called a high-resolution diagram or a peak estimator.

The high-resolution diagrams provide DOA parameter estimates. They generally do not allow the estimation of the sound power versus azimuth and elevation, especially the respective signal and noise power distribution.

5.4.3 High-Resolution Diagrams

Many of the known high resolution peak estimators can be motivated by certain properties of the covariance matrix

$$\mathbf{c}_{\mathbf{xx}} = \mathbf{A}\,\mathbf{c}_{\mathbf{ss}}\mathbf{A}^{H} + \sigma_{\mathbf{u}}^{2}\mathbf{I}$$

of the vector valued array output

$$\mathbf{x}(t) = \left(x_1(t), \ldots, x_N(t)\right)^T.$$

Here, $\mathbf{c}_{\mathbf{ss}}$ is the covariance matrix of the source signals

$$\mathbf{s}(t) = \left(s_1(t), \ldots, s_M(t)\right)^T$$

which is not necessarily diagonal, $\mathbf{A}$ is an $N \times M$ matrix with

$$\mathbf{A} = \left(\mathbf{a}(\mathbf{k}_1), \ldots, \mathbf{a}(\mathbf{k}_M)\right) = \left(\tilde{\mathbf{a}}(\varphi_1, \vartheta_1), \ldots, \tilde{\mathbf{a}}(\varphi_M, \vartheta_M)\right),$$

$\sigma_{\mathbf{u}}^2$ is the noise power and $\mathbf{I}$ denotes the identity matrix.

We assume $M < N$ and both, $\mathbf{c_{ss}}$ and $\mathbf{A}$ are of full rank M, i.e. the signals are not fully coherent, and the array is suitably designed for the wave vectors $\mathbf{k}$ resp. angles (φ, ϑ) of interest.

Now, let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues and $\mathbf{v}_1, \ldots, \mathbf{v}_N$ the corresponding orthonormal eigenvectors of the nonnegative definite, Hermitian matrix $\mathbf{c_{xx}}$, i.e.

$$\mathbf{c_{xx}} \mathbf{v}_n = \lambda_n \mathbf{v}_n, \quad n = 1, \ldots, N,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$. Then we can state that

$$\lambda_{M+1} = \cdots = \lambda_N = \sigma_{\mathbf{u}}^2$$

and that because of $\lambda_M > \lambda_{M+1}$ the corresponding eigenvectors

$$\mathbf{v}_{M+1}, \ldots, \mathbf{v}_N \qquad \text{(spanning the only noise subspace)}$$

are orthogonal to the column vectors of

$$\mathbf{A} = \left(\mathbf{a}(\mathbf{k}_1),\ldots,\mathbf{a}(\mathbf{k}_M)\right) = \left(\tilde{\mathbf{a}}(\varphi_1,\vartheta_1),\ldots,\tilde{\mathbf{a}}(\varphi_M,\vartheta_M)\right)$$

which are often called steering vectors.

Furthermore, the eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_M$ (spanning the signal subspace) can be represented as linear combinations of the steering vectors and vice versa.

Proof sketch:

$$\operatorname{rank}(\mathbf{A}) = M,\ \operatorname{rank}(\mathbf{c}_{\mathbf{ss}}) = M \ \Rightarrow\ \operatorname{rank}(\mathbf{A}\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H) = M < N$$

Hence, $\mathbf{A}\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H$ is a Hermitian positive semidefinite matrix that possesses the eigenvalue/eigenvector decomposition

$$\mathbf{A}\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H = \mathbf{V}\operatorname{diag}(\mu_1,\ldots,\mu_M,0,\ldots,0)\,\mathbf{V}^H$$

which by means of the following decomposition

$$\mathbf{V} = (\mathbf{V}_s, \mathbf{V}_n) = (\underbrace{\mathbf{v}_1, \ldots, \mathbf{v}_M}_{\mathbf{V}_s}, \underbrace{\mathbf{v}_{M+1}, \ldots, \mathbf{v}_N}_{\mathbf{V}_n}),$$

can be simplified to

$$\mathbf{A}\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H = \mathbf{V}_s \operatorname{diag}(\mu_1, \ldots, \mu_M)\mathbf{V}_s^H = \sum_{m=1}^{M} \mu_m \mathbf{v}_m \mathbf{v}_m^H .$$

Substitution in $\mathbf{c}_{\mathbf{xx}}$ provides

$$\mathbf{c}_{\mathbf{xx}} = \sum_{m=1}^{M} \mu_m \mathbf{v}_m \mathbf{v}_m^H + \sigma_{\mathbf{u}}^2 \mathbf{I}$$

and subsequently the eigenvalue/eigenvector decomposition

$$\mathbf{c}_{\mathbf{xx}}\mathbf{v}_i = \lambda_i \mathbf{v}_i \quad \text{with} \quad \lambda_i = \begin{cases} \mu_i + \sigma_{\mathbf{u}}^2 & i = 1, \ldots, M \\ \sigma_{\mathbf{u}}^2 & i = M+1, \ldots, N \end{cases}.$$

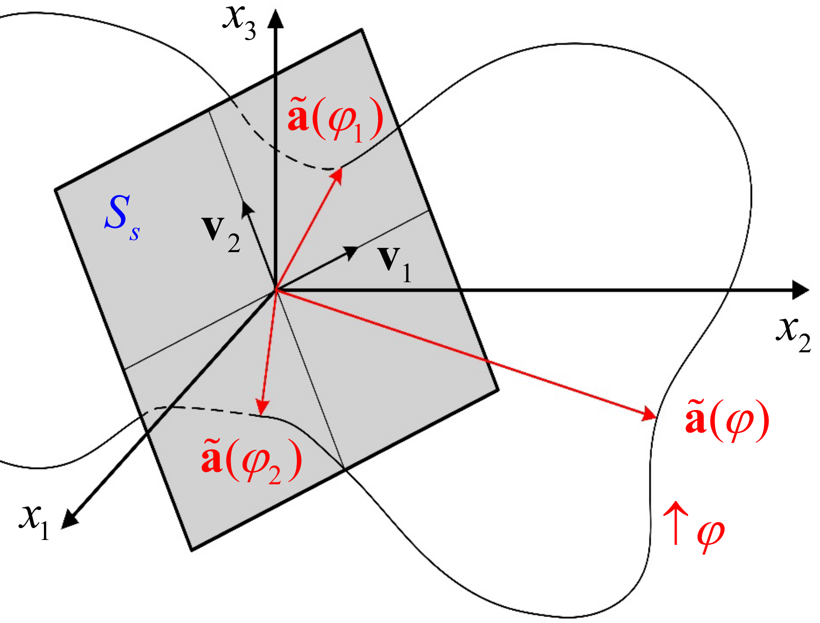
by obvious deductive reasoning.

Furthermore,

$$S_s = \mathrm{span}(\mathbf{A}) = \mathrm{span}(\mathbf{V}_s)$$

$$S_n = \mathrm{span}(\mathbf{V}_n)$$

define the signal and only noise subspace, respectively, with $S_s \perp S_n$.



The properties of the covariance matrix $\mathbf{c}_{\mathbf{xx}}$ can be exploited to construct high-resolution diagrams if the estimate $\hat{\mathbf{c}}_{\mathbf{xx}}$ possesses approximately the same properties.

Let $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \ldots \geq \hat{\lambda}_N$ and $\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \ldots, \hat{\mathbf{v}}_N$ denote the eigenvalues and corresponding eigenvectors of $\hat{\mathbf{c}}_{\mathbf{xx}}$, i.e.

$$\hat{\mathbf{c}}_{\mathbf{xx}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{x}(lT_S)\,\mathbf{x}^H(lT_S) = \sum_{n=1}^{N}\hat{\lambda}_n\hat{\mathbf{v}}_n\hat{\mathbf{v}}_n^H,$$

then under certain regularity conditions on can show, that the eigenvalues and eigenvectors of $\mathbf{c}_{\mathbf{xx}}$ can be consistently estimated by the eigenvalues and eigenvectors of the sample covariance matrix, i.e.

$$\hat{\lambda}_n \xrightarrow[L\to\infty]{} \lambda_n \quad \text{and} \quad \hat{\mathbf{v}}_n \xrightarrow[L\to\infty]{} \mathbf{v}_n \quad \text{for} \quad n = 1,\ldots,N.$$

Exploiting the eigenvalue decomposition of $\hat{\mathbf{c}}_{\mathbf{xx}}$, the classical beamformer can be written as

$$q_{CB}(\mathbf{k}) = \mathbf{a}^H(\mathbf{k})\,\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{a}(\mathbf{k}) = \sum_{n=1}^{N}\hat{\lambda}_n\left|\hat{\mathbf{v}}_n^H\mathbf{a}(\mathbf{k})\right|^2$$

$$\tilde{q}_{CB}(\varphi,\vartheta) = \tilde{\mathbf{a}}^H(\varphi,\vartheta)\,\hat{\mathbf{c}}_{\mathbf{xx}}\tilde{\mathbf{a}}(\varphi,\vartheta) = \sum_{n=1}^{N}\hat{\lambda}_n\left|\hat{\mathbf{v}}_n^H\tilde{\mathbf{a}}(\varphi,\vartheta)\right|^2.$$

Example:

For a single source embedded in spatially white noise the covariance matrix of the receiver outputs is given by

$$\mathbf{c}_{\mathbf{xx}} = \sigma_s^2 \mathbf{a}(\mathbf{k}_s)\mathbf{a}^H(\mathbf{k}_s) + \sigma_{\mathbf{u}}^2 \mathbf{I} \quad \text{with} \quad \mathbf{a}^H(\mathbf{k}_s)\mathbf{a}(\mathbf{k}_s) = N.$$

Multiplication of $\mathbf{c}_{\mathbf{xx}}$ by $\mathbf{a}(\mathbf{k}_s)$ from the right provides

$$\mathbf{c}_{\mathbf{xx}}\mathbf{a}(\mathbf{k}_s) = \sigma_s^2 \mathbf{a}(\mathbf{k}_s)\mathbf{a}^H(\mathbf{k}_s)\mathbf{a}(\mathbf{k}_s) + \sigma_{\mathbf{u}}^2 \mathbf{a}(\mathbf{k}_s) = \left(N\sigma_s^2 + \sigma_{\mathbf{u}}^2\right)\mathbf{a}(\mathbf{k}_s).$$

Comparison with the eigenvalue/eigenvector decomposition mentioned on pp. 59-61 we can assert that

$$\lambda_1 = N\sigma_s^2 + \sigma_{\mathbf{u}}^2 > \lambda_2 = \cdots = \lambda_N = \sigma_{\mathbf{u}}^2, \quad \mathbf{v}_1 = \mathbf{a}(\mathbf{k}_s)\big/\sqrt{N}$$

and that the remaining eigenvectors can be selected such that the $\mathbf{v}_n, n = 1, \ldots, N$ are forming an orthonormal basis.

From the well known result

$$\lambda_1 = \max_{\mathbf{z}\in\mathbb{C}^N:|\mathbf{z}|=1} \mathbf{z}^H \mathbf{c}_{\mathbf{xx}} \mathbf{z} \quad \text{and} \quad \mathbf{v}_1 = \arg\max_{\mathbf{z}\in\mathbb{C}^N:|\mathbf{z}|=1} \mathbf{z}^H \mathbf{c}_{\mathbf{xx}} \mathbf{z}$$

and the fact that $\mathbf{v}_1 = \mathbf{a}(\mathbf{k}_s)/\sqrt{N}$, we can deduce

$$\lambda_1 = \max_{\mathbf{k}} \frac{\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{xx}}\mathbf{a}(\mathbf{k})}{N} \quad \text{and} \quad \mathbf{k}_s = \arg\max_{\mathbf{k}} \frac{\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{xx}}\mathbf{a}(\mathbf{k})}{N}.$$

Hence, the classical beamformer provides for a single source embedded in spatial white noise the consistent estimate

$$\hat{\mathbf{k}}_s = \arg\max_{\mathbf{k}} q_{CB}(\mathbf{k}) = \arg\max_{\mathbf{k}} \mathbf{a}^H(\mathbf{k})\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{a}(\mathbf{k})$$

$$= \arg\max_{\mathbf{k}} \sum_{n=1}^{N} \hat{\lambda}_n \left|\hat{\mathbf{v}}_n^H \mathbf{a}(\mathbf{k})\right|^2 \xrightarrow[L\to\infty]{} \mathbf{k}_s$$

Mathematical Supplement

Constraint Optimization / Equality Constraints

Find the optimum of $y = f(\mathbf{x})$ subject to $h(\mathbf{x}) = 0$, where

$$\mathbf{x} \in D(f) \subset \mathbb{R}^n \text{ and } \nabla h(\mathbf{x}) \neq 0, \forall \mathbf{x} \in D(f).$$

Let $\mathbf{x}_0 \in D(f)$ be a solution of the constraint optimization problem then a certain $\alpha \in \mathbb{R}$ exists such that

$$\nabla f(\mathbf{x}_0) + \alpha \nabla h(\mathbf{x}_0) = \mathbf{0}.$$

Proof sketch:

$$M = \{\mathbf{x} \in D(f):\ h(\mathbf{x}) = 0\}$$

defines a $(n-1)$-dimensional surface in $D(f)$. Let

$$\mathbf{x} \in M,\ \ \mathbf{x} + \Delta\mathbf{x} \in M,\ \text{ i.e. } h(\mathbf{x}) = h(\mathbf{x} + \Delta\mathbf{x}) = 0$$

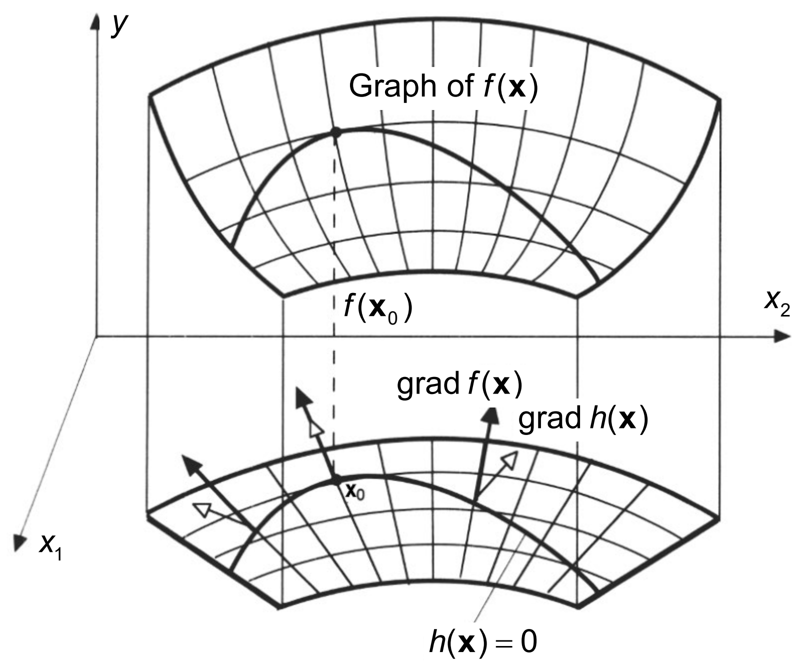
and

$$h(\mathbf{x}+\Delta\mathbf{x}) = h(\mathbf{x}) + \Delta\mathbf{x}^T \nabla h(\mathbf{x}) + o\left(|\Delta\mathbf{x}|\right)$$

be the Taylor expansion around **x**, then, due to

$$\Delta\mathbf{x}^T \nabla h(\mathbf{x}) = o\left(|\Delta\mathbf{x}|\right)$$

$$\Rightarrow \frac{\Delta\mathbf{x}^T}{|\Delta\mathbf{x}|} \nabla h(\mathbf{x}) \xrightarrow[|\Delta\mathbf{x}|\to 0]{} 0$$



we can conclude that

$$\forall \mathbf{x} \in M,\ \nabla h(\mathbf{x})$$

is normal to the surface.

Furthermore, $\nabla f(\mathbf{x}_0)$ is also orthogonal to the surface because otherwise we could increase the value of $f(\mathbf{x})$ by moving a short distance along the surface.

<u>Example:</u>

$$f(x_1,x_2)=x_1^2+x_2^2+3,\;\; h(x_1,x_2)=x_1^2+x_2-2=0,\;\; \mathbf{x}=(x_1,x_2)^T\in\mathbb{R}^2$$

The Lagrange-Function can be expressed by

$$L(x_1,x_2,\alpha)=f(x_1,x_2)+\alpha h(x_1,x_2)=x_1^2+x_2^2+3+\alpha(x_1^2+x_2-2).$$

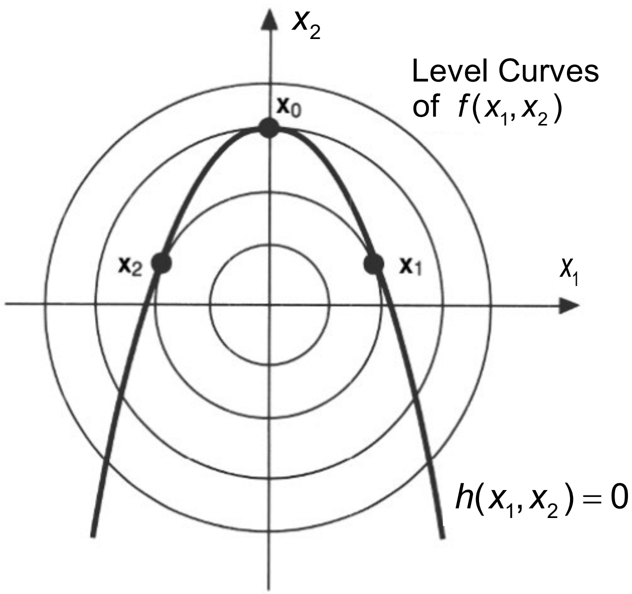
Hence, the necessary conditions given by

$$\nabla L(x_1,x_2,\alpha)=\nabla f(x_1,x_2)+\alpha\nabla h(x_1,x_2)=2\begin{pmatrix}x_1\\x_2\end{pmatrix}+\alpha\begin{pmatrix}2x_1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\frac{\partial}{\partial\alpha}L(x_1,x_2,\alpha)=h(x_1,x_2)=x_1^2+x_2-2=0$$

provide the three equations that lead after the following considerations to the possible extrema.

$$
\begin{array}{llll}
1) & x_1 = 0: & 2x_2 = -\alpha, & \\
 & & x_2 = 2, & \Rightarrow \alpha = -4 \\
2) & x_1 \neq 0: & x_1(1+\alpha) = 0 & \Rightarrow \alpha = -1 \\
 & & 2x_2 - 1 = 0 & \Rightarrow x_2 = 1/2 \\
 & & x_1^2 - 3/2 = 0 & \Rightarrow x_1 = \pm\sqrt{3/2} = \pm\sqrt{6}/2
\end{array}
$$



Possible Extrema

$$\mathbf{x}_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \mathbf{x}_{1,2} = \begin{pmatrix} \pm\sqrt{6}/2 \\ 1/2 \end{pmatrix}$$

Maximum: $f(\mathbf{x}_0 + \Delta\mathbf{x}) \leq f(\mathbf{x}_0)$

Minima: $f(\mathbf{x}_1 + \Delta\mathbf{x}) \geq f(\mathbf{x}_1)$

$f(\mathbf{x}_2 + \Delta\mathbf{x}) \geq f(\mathbf{x}_2)$

Real valued function of a complex variable / complex derivatives

$$f:\mathbb{C}\rightarrow\mathbb{R}\;\Rightarrow\; f(z)=f(x+jy)=\tilde{f}(x,y)=g(z,z^*)$$

Example:

$$f(z)=zz^*=g(z,z^*)=(x+jy)(x-jy)=x^2+y^2=\tilde{f}(x,y)$$

Now one can proof that both

$$\frac{\partial g(z,z^*)}{\partial z}=0,\;\text{ where } z^* \text{ is treated as constant}$$

and

$$\frac{\partial g(z,z^*)}{\partial z^*}=0,\;\text{ where } z \text{ is treated as constant}$$

provide a necessary and sufficient condition for a stationary point of $f(z)$.

$$\left.\begin{array}{l}
\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} \partial \tilde{f}/\partial x \\ \partial \tilde{f}/\partial y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
\dfrac{\partial g(z,z^*)}{\partial z} = z^* = x - jy = 0 \\
\dfrac{\partial g(z,z^*)}{\partial z^*} = z = x + jy = 0
\end{array}\right\} \Rightarrow \ x = 0,\ y = 0$$

Real valued function of complex variables / complex gradients

$$f:\ \mathbb{C}^n \to \mathbb{R} \ \Rightarrow \ f(\mathbf{z}) = f(\mathbf{x} + j\mathbf{y}) = \tilde{f}(\mathbf{x},\mathbf{y}) = g(\mathbf{z},\mathbf{z}^H)$$

Example:

$$f(\mathbf{z}) = \mathbf{z}^H\mathbf{z} = g(\mathbf{z},\mathbf{z}^H) = \mathbf{x}^T\mathbf{x} + \mathbf{y}^T\mathbf{y} = \tilde{f}(\mathbf{x},\mathbf{y})$$

Here one can also proof that both

$$\nabla_{\mathbf{z}}\, g(\mathbf{z},\mathbf{z}^H) = \mathbf{0} \text{ and } \nabla_{\mathbf{z}^H}\, g(\mathbf{z},\mathbf{z}^H) = \mathbf{0}$$

can serve as a necessary and sufficient condition for a stationary point of $f(\mathbf{z})$, where $\mathbf{z}^H$ and $\mathbf{z}$ are treated as constant respectively.

$$\left.\begin{aligned} &\begin{pmatrix} \nabla_{\mathbf{x}} f \\ \nabla_{\mathbf{y}} f \end{pmatrix} = \begin{pmatrix} \nabla_{\mathbf{x}} \tilde{f} \\ \nabla_{\mathbf{y}} \tilde{f} \end{pmatrix} = \begin{pmatrix} 2\mathbf{x} \\ 2\mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \\ &\nabla_{\mathbf{z}} g(\mathbf{z},\mathbf{z}^H) = \mathbf{z}^* = \mathbf{x} - j\mathbf{y} = \mathbf{0} \\ &\nabla_{\mathbf{z}^H} g(\mathbf{z},\mathbf{z}^H) = \mathbf{z} = \mathbf{x} + j\mathbf{y} = \mathbf{0} \end{aligned}\right\} \Rightarrow \mathbf{x} = 0, \mathbf{y} = 0$$

Minimum Power Distortionless Response Beamformer

The sensor outputs are weighted by a vector **w** to produce the beamformer output signal

$$y(t) = \mathbf{w}^H \mathbf{x}(t).$$

The minimum power distortionless response (MPDR) diagram is derived by finding the vector **w** which minimizes the power of the beamformer output signal, i.e.

$$\mathrm{E}\left|y(t)\right|^2 = \mathbf{w}^H \mathbf{c}_{\mathbf{xx}} \mathbf{w},$$

subject to the constraint that

$$\mathbf{w}^H \mathbf{a}(\mathbf{k}) = 1 \;\text{ resp. }\; \mathbf{w}^H \tilde{\mathbf{a}}(\varphi, \vartheta) = 1.$$

Thus, the MPDR beamformer ensures that a signal incident on the array from direction **k** resp. angle (φ,ϑ) is passed to the output undistorted, while simultaneously contributions due to noise and interfering signals arriving from other directions are minimized.

By introducing the Lagrange multiplier α, the linearly constrained minimization problem

$$\min_{\mathbf{w},\mathbf{w}^H\mathbf{a}(\mathbf{k})=1}\left(\mathbf{w}^H\mathbf{c}_{\mathbf{xx}}\mathbf{w}\right)$$

can be transferred to the unconstrained problem

$$\min_{\mathbf{w},\alpha}\left(\mathbf{w}^H\mathbf{c}_{\mathbf{xx}}\mathbf{w}+\alpha\left(\mathbf{w}^H\mathbf{a}(\mathbf{k})-1\right)+\alpha^*\left(\mathbf{a}^H(\mathbf{k})\mathbf{w}-1\right)\right)$$

resp.

$$\min_{\mathbf{w},\alpha}\left(\mathbf{w}^H\mathbf{c}_{\mathbf{xx}}\mathbf{w}+\alpha\left(\mathbf{w}^H\tilde{\mathbf{a}}(\varphi,\vartheta)-1\right)+\alpha^*\left(\tilde{\mathbf{a}}^H(\varphi,\vartheta)\mathbf{w}-1\right)\right).$$

Hence, the Lagrange Function

$$L(\mathbf{w},\mathbf{w}^H,\alpha,\alpha^*)=\mathbf{w}^H\mathbf{c}_{\mathbf{xx}}\mathbf{w}+\alpha\left(\mathbf{w}^H\mathbf{a}(\mathbf{k})-1\right)+\alpha^*\left(\mathbf{a}^H(\mathbf{k})\mathbf{w}-1\right).$$

has to be minimized. The necessary condition

$$\nabla_{\mathbf{w}^H}L(\mathbf{w},\mathbf{w}^H,\alpha,\alpha^*)=(\nabla_{\mathbf{w}^H}\mathbf{w}^H)\,\mathbf{c}_{\mathbf{xx}}\mathbf{w}+\alpha(\nabla_{\mathbf{w}^H}\mathbf{w}^H)\,\mathbf{a}(\mathbf{k})$$

$$=\mathbf{I}\,\mathbf{c}_{\mathbf{xx}}\mathbf{w}+\alpha\,\mathbf{I}\,\mathbf{a}(\mathbf{k})=\mathbf{c}_{\mathbf{xx}}\mathbf{w}+\alpha\,\mathbf{a}(\mathbf{k})=0$$

provides $\mathbf{w}=-\alpha\,\mathbf{c}_{\mathbf{xx}}^{-1}\mathbf{a}(\mathbf{k})$ which inserted in

$$\frac{\partial}{\partial\alpha^*}L(\mathbf{w},\mathbf{w}^H,\alpha,\alpha^*)=\mathbf{a}^H(\mathbf{k})\,\mathbf{w}-1=0$$

leads us to

$$-\alpha\,\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{xx}}^{-1}\mathbf{a}(\mathbf{k})-1=0\;\Rightarrow\;\alpha=-1\Big/\left(\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{xx}}^{-1}\mathbf{a}(\mathbf{k})\right)$$

and consequently to the solution for **w** given by

$$\mathbf{w} = \frac{\mathbf{c}_{\mathbf{xx}}^{-1}\mathbf{a}(\mathbf{k})}{\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{xx}}^{-1}\mathbf{a}(\mathbf{k})} \quad \text{resp.} \quad \mathbf{w} = \frac{\mathbf{c}_{\mathbf{xx}}^{-1}\tilde{\mathbf{a}}(\varphi,\vartheta)}{\tilde{\mathbf{a}}^H(\varphi,\vartheta)\mathbf{c}_{\mathbf{xx}}^{-1}\tilde{\mathbf{a}}(\varphi,\vartheta)}.$$

Inserting this solution into $\mathbf{w}^H\mathbf{c}_{\mathbf{xx}}\mathbf{w}$ and replacing $\mathbf{c}_{\mathbf{xx}}$ by its consistent estimate $\hat{\mathbf{c}}_{\mathbf{xx}}$ the MPDR / Capon beamformer can be expressed as follows.

$$q_C(\mathbf{k}) = \frac{1}{\mathbf{a}^H(\mathbf{k})\hat{\mathbf{c}}_{\mathbf{xx}}^{-1}\mathbf{a}(\mathbf{k})} = \left(\sum_{n=1}^{N}\hat{\lambda}_n^{-1}\left|\hat{\mathbf{v}}_n^H\mathbf{a}(\mathbf{k})\right|^2\right)^{-1}$$

$$\tilde{q}_C(\varphi,\vartheta) = \frac{1}{\tilde{\mathbf{a}}^H(\varphi,\vartheta)\hat{\mathbf{c}}_{\mathbf{xx}}^{-1}\tilde{\mathbf{a}}(\varphi,\vartheta)} = \left(\sum_{n=1}^{N}\hat{\lambda}_n^{-1}\left|\hat{\mathbf{v}}_n^H\tilde{\mathbf{a}}(\varphi,\vartheta)\right|^2\right)^{-1}$$

Estimate updating: *(inverse covariance matrix)*

$$\hat{\mathbf{c}}_{\mathbf{xx},l} = \eta\left(\hat{\mathbf{c}}_{\mathbf{xx},l-1} + \gamma\mathbf{x}(lT_s)\,\mathbf{x}^H(lT_s)\right)$$

$$\text{with} \begin{cases} \gamma = 1/l, \quad \eta = l/(l+1) & \text{for } \hat{\mathbf{c}}_{\mathbf{xx},l} = \hat{\mathbf{c}}^G_{\mathbf{xx},l} \\ \gamma = \beta/(1-\beta), \quad \eta = 1-\beta & \text{for } \hat{\mathbf{c}}_{\mathbf{xx},l} = \hat{\mathbf{c}}^E_{\mathbf{xx},l} \end{cases}$$

Assuming that $\hat{\mathbf{c}}_{\mathbf{xx},l}$ is invertible, which implies that $l \geq N$ is required, and exploiting the Matrix Inversion Lemma

$$(\mathbf{A}+\mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1}+\mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$$

one can derive the computational efficient recursion

$$\hat{\mathbf{c}}^{-1}_{\mathbf{xx},l} = \frac{1}{\eta}\left(\hat{\mathbf{c}}^{-1}_{\mathbf{xx},l-1} - \frac{\hat{\mathbf{c}}^{-1}_{\mathbf{xx},l-1}\mathbf{x}(lT_s)\,\mathbf{x}^H(lT_s)\hat{\mathbf{c}}^{-1}_{\mathbf{xx},l-1}}{1/\gamma + \mathbf{x}^H(lT_s)\,\hat{\mathbf{c}}^{-1}_{\mathbf{xx},l-1}\mathbf{x}(lT_s)}\right).$$

Simulation Results of MPDR / Capon Beamforming

$\varphi_1 = -7°,\ \varphi_2 = -2°,\ \varphi_3 = 10°$

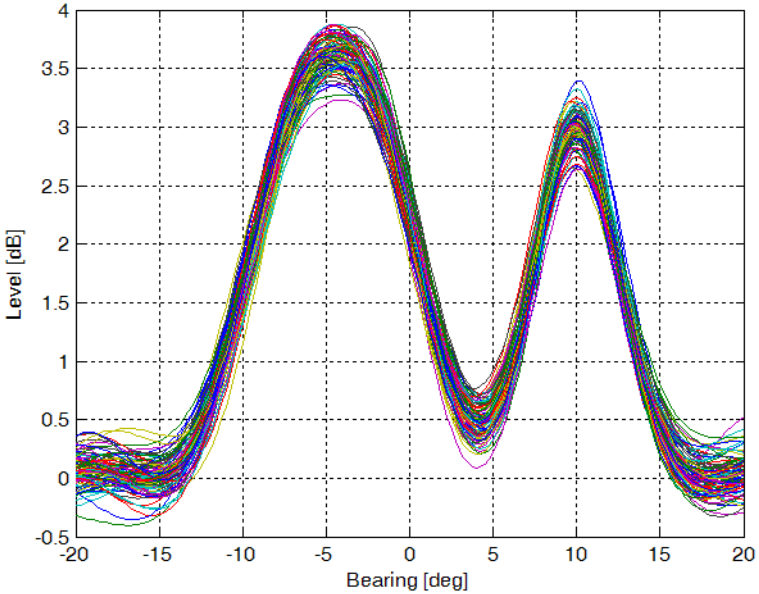
$L = 1000,\ \ N = 15$

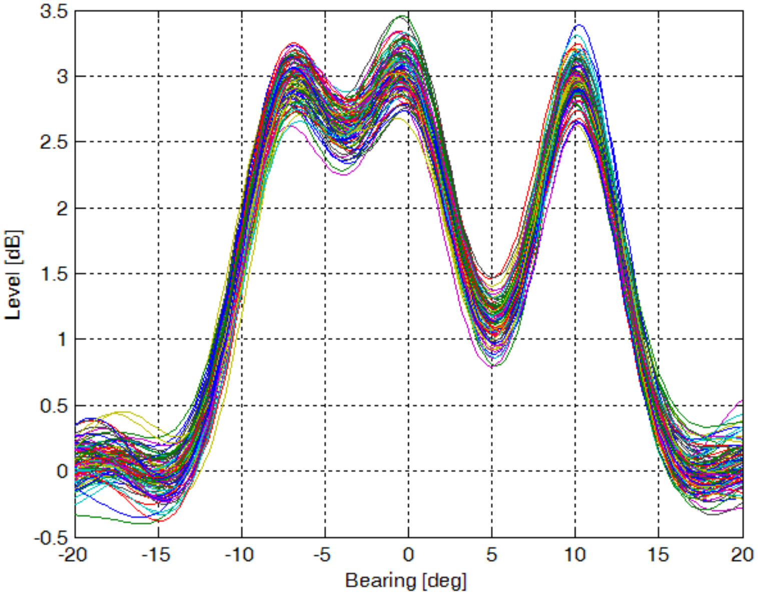
$SNR = -10\log_{10}(N)\,\mathrm{dB}$

$\varphi_1 = -7°,\ \varphi_2 = 0°,\ \varphi_3 = 10°$

$L = 1000,\ \ N = 15$

$SNR = -10\log_{10}(N)\,\mathrm{dB}$





Remarks: MPDR / Capon Beamforming

Assets

- Number of sources must not be known
- Moderate computational effort (efficient inversion)
- Fair resolving power

Drawbacks

- Narrowband / generalization to broadband not straight forward
- Estimates signal + noise power only
- Matrix inversion may cause problems
- Peak finding can be difficult

In case that the noise covariance matrix is known the minimum variance distortionless response (MVDR) beamformer can be derived by finding the vector **w** that minimizes the noise power, i.e.

$$\mathrm{E}\left|\mathbf{w}^H\mathbf{u}(t)\right|^2 = \mathbf{w}^H\mathbf{c}_{\mathbf{uu}}\mathbf{w},$$

subject to the constraint

$$\mathbf{w}^H\mathbf{a}(\mathbf{k}) = 1 \quad \text{resp.} \quad \mathbf{w}^H\tilde{\mathbf{a}}(\varphi,\vartheta) = 1.$$

Analogous to the MPDR the solution

$$\mathbf{w} = \frac{\mathbf{c}_{\mathbf{uu}}^{-1}\mathbf{a}(\mathbf{k})}{\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{uu}}^{-1}\mathbf{a}(\mathbf{k})} \quad \text{resp.} \quad \mathbf{w} = \frac{\mathbf{c}_{\mathbf{uu}}^{-1}\tilde{\mathbf{a}}(\varphi,\vartheta)}{\tilde{\mathbf{a}}^H(\varphi,\vartheta)\mathbf{c}_{\mathbf{uu}}^{-1}\tilde{\mathbf{a}}(\varphi,\vartheta)}$$

can be derived for the MVDR.

Now, if the noise covariance matrix is given by

$$\mathbf{c_{uu}} = \mathbf{A}_I \, \mathbf{c}_{\mathbf{s}_I\mathbf{s}_I} \mathbf{A}_I^H + \sigma_{\mathbf{u}}^2 \mathbf{I}, \quad \mathbf{A}_I = \left(\mathbf{a}(\mathbf{k}_{I,1}), \ldots, \mathbf{a}(\mathbf{k}_{I,J})\right)$$

the application of the Matrix Inversion Lemma

$$(\mathbf{A} + \mathbf{BCD})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}(\mathbf{C}^{-1} + \mathbf{DA}^{-1}\mathbf{B})^{-1}\mathbf{DA}^{-1}$$

provides the inverse noise covariance matrix

$$\mathbf{c}_{\mathbf{uu}}^{-1} = \frac{1}{\sigma_{\mathbf{u}}^2}\mathbf{I} - \frac{1}{\sigma_{\mathbf{u}}^2}\mathbf{A}_I \left(\mathbf{c}_{\mathbf{s}_I\mathbf{s}_I}^{-1} + \frac{1}{\sigma_{\mathbf{u}}^2}\mathbf{A}_I^H \mathbf{A}_I \right)^{-1} \mathbf{A}_I^H \frac{1}{\sigma_{\mathbf{u}}^2}$$

$$= \frac{1}{\sigma_{\mathbf{u}}^2}\left[\mathbf{I} - \mathbf{A}_I \left(\sigma_{\mathbf{u}}^2 \mathbf{c}_{\mathbf{s}_I\mathbf{s}_I}^{-1} + \mathbf{A}_I^H \mathbf{A}_I \right)^{-1} \mathbf{A}_I^H \right],$$

where $\mathbf{c}_{\mathbf{s}_I\mathbf{s}_I}$ and $\mathbf{k}_{I,j}$ denote the covariance matrix and the wave number vectors of the interference and $\sigma_{\mathbf{u}}^2$ the noise variance.

Example: *(a single interference)*

$$\mathbf{c}_{\mathbf{uu}} = \sigma_{\mathbf{s}_I}^2 \mathbf{a}(\mathbf{k}_I)\mathbf{a}^H(\mathbf{k}_I) + \sigma_{\mathbf{u}}^2 \mathbf{I}$$

$$\Rightarrow \mathbf{c}_{\mathbf{uu}}^{-1} = \frac{1}{\sigma_{\mathbf{u}}^2}\left(\mathbf{I} - \mathbf{a}(\mathbf{k}_I)\left(\sigma_{\mathbf{u}}^2/\sigma_{\mathbf{s}_I}^2 + N\right)^{-1}\mathbf{a}^H(\mathbf{k}_I)\right)$$

For $\sigma_{\mathbf{u}}^2/\sigma_{\mathbf{s}_I}^2 \ll \mathbf{a}^H(\mathbf{k}_I)\mathbf{a}(\mathbf{k}_I) = N$ we can approximately write

$$\mathbf{c}_{\mathbf{uu}}^{-1} = \frac{1}{\sigma_{\mathbf{u}}^2}\left(\mathbf{I} - \mathbf{a}(\mathbf{k}_I)\left(\mathbf{a}^H(\mathbf{k}_I)\mathbf{a}(\mathbf{k}_I)\right)^{-1}\mathbf{a}^H(\mathbf{k}_I)\right) = \frac{1}{\sigma_{\mathbf{u}}^2}(\mathbf{I} - \mathbf{P}_I) = \frac{1}{\sigma_{\mathbf{u}}^2}\mathbf{P}_I^{\perp},$$

where the Hermitian and idempotent matrices

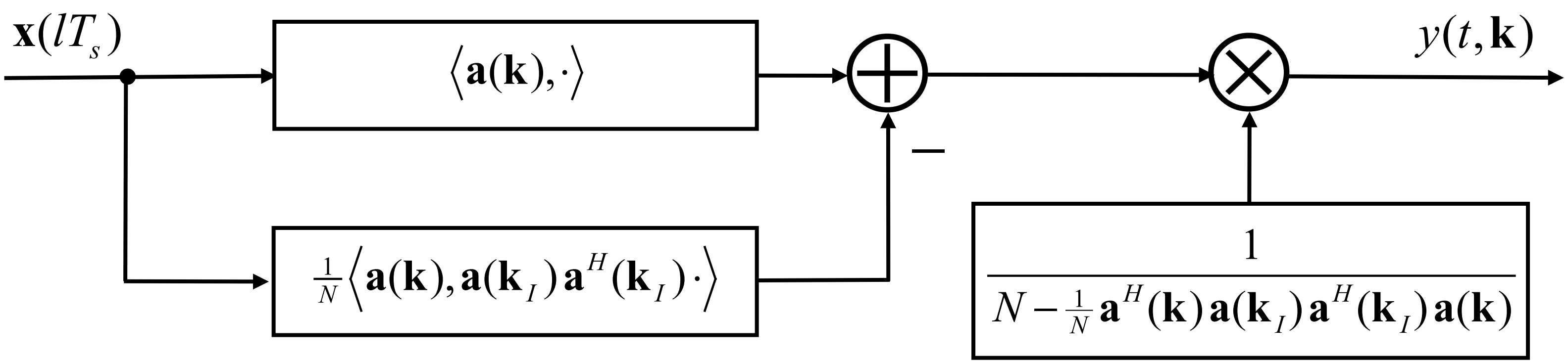
$$\mathbf{P}_I = \mathbf{a}(\mathbf{k}_I)\left(\mathbf{a}^H(\mathbf{k}_I)\mathbf{a}(\mathbf{k}_I)\right)^{-1}\mathbf{a}^H(\mathbf{k}_I) = \mathbf{a}(\mathbf{k}_I)\mathbf{a}^H(\mathbf{k}_I)/N$$

and $\mathbf{P}_I^{\perp} = \mathbf{I} - \mathbf{P}_I = \mathbf{I} - \mathbf{a}(\mathbf{k}_I)\mathbf{a}^H(\mathbf{k}_I)/N$

denote projection matrices, that project a N-dimensional vector onto the 1-dimensional interference and $(N-1)$-dimensional only noise subspace respectively. Finally, we obtain

$$\mathbf{w} = \frac{\mathbf{c}_{\mathbf{uu}}^{-1}\mathbf{a}(\mathbf{k})}{\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{uu}}^{-1}\mathbf{a}(\mathbf{k})} = \frac{\mathbf{P}_I^{\perp}\mathbf{a}(\mathbf{k})}{\mathbf{a}^H(\mathbf{k})\mathbf{P}_I^{\perp}\mathbf{a}(\mathbf{k})}$$

$$= \frac{\mathbf{a}(\mathbf{k}) - \mathbf{P}_I\mathbf{a}(\mathbf{k})}{N - \mathbf{a}^H(\mathbf{k})\mathbf{P}_I\mathbf{a}(\mathbf{k})} = \frac{\mathbf{a}(\mathbf{k}) - \frac{1}{N}\mathbf{a}(\mathbf{k}_I)\mathbf{a}^H(\mathbf{k}_I)\mathbf{a}(\mathbf{k})}{N - \frac{1}{N}\mathbf{a}^H(\mathbf{k})\mathbf{a}(\mathbf{k}_I)\mathbf{a}^H(\mathbf{k}_I)\mathbf{a}(\mathbf{k})}.$$



Example: *(multiple uncorrelated interferences)*

$$\mathbf{c}_{\mathbf{uu}} = \mathbf{A}_I \,\mathrm{diag}\left(\sigma^2_{\mathbf{s}_{I,1}},\ldots,\sigma^2_{\mathbf{s}_{I,J}}\right)\mathbf{A}_I^H + \sigma^2_{\mathbf{u}}\mathbf{I}$$

$$\Rightarrow\ \mathbf{c}_{\mathbf{uu}}^{-1} = \frac{1}{\sigma^2_{\mathbf{u}}}\left(\mathbf{I} - \mathbf{A}_I\left(\mathrm{diag}\left(\sigma^2_{\mathbf{u}}/\sigma^2_{\mathbf{s}_{I,1}},\ldots,\sigma^2_{\mathbf{u}}/\sigma^2_{\mathbf{s}_{I,J}}\right) + \mathbf{A}_I^H\mathbf{A}_I\right)^{-1}\mathbf{A}_I^H\right)$$

For $\sigma^2_{\mathbf{u}}/\sigma^2_{\mathbf{s}_{I,j}} \ll \mathbf{a}^H(\mathbf{k}_{I,j})\mathbf{a}(\mathbf{k}_{I,j}) = N,\ \forall j = 1,\ldots,J$ we can again approximately write

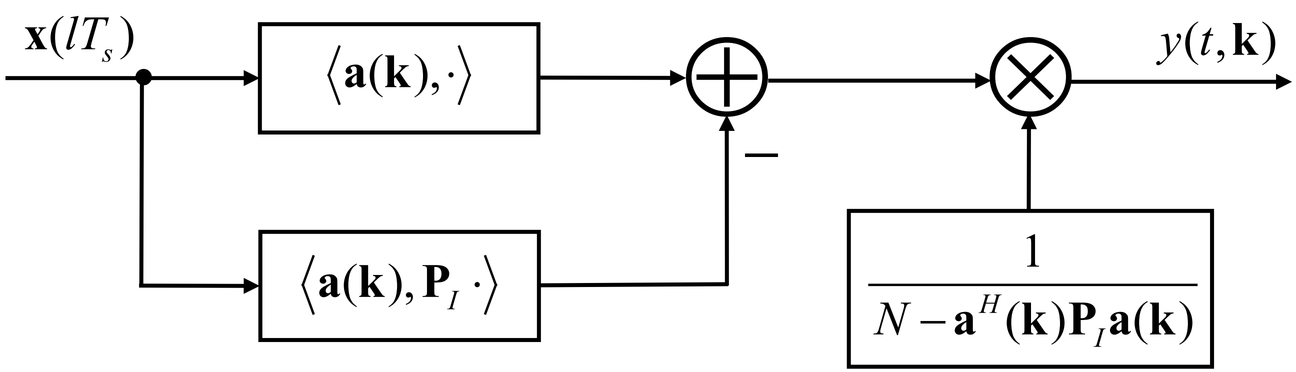
$$\mathbf{c}_{\mathbf{uu}}^{-1} = \frac{1}{\sigma^2_{\mathbf{u}}}\left(\mathbf{I} - \mathbf{A}_I\left(\mathbf{A}_I^H\mathbf{A}_I\right)^{-1}\mathbf{A}_I^H\right) = \frac{1}{\sigma^2_{\mathbf{u}}}\left(\mathbf{I} - \mathbf{P}_I\right) = \frac{1}{\sigma^2_{\mathbf{u}}}\mathbf{P}_I^{\perp},$$

where the Hermitian and idempotent matrices

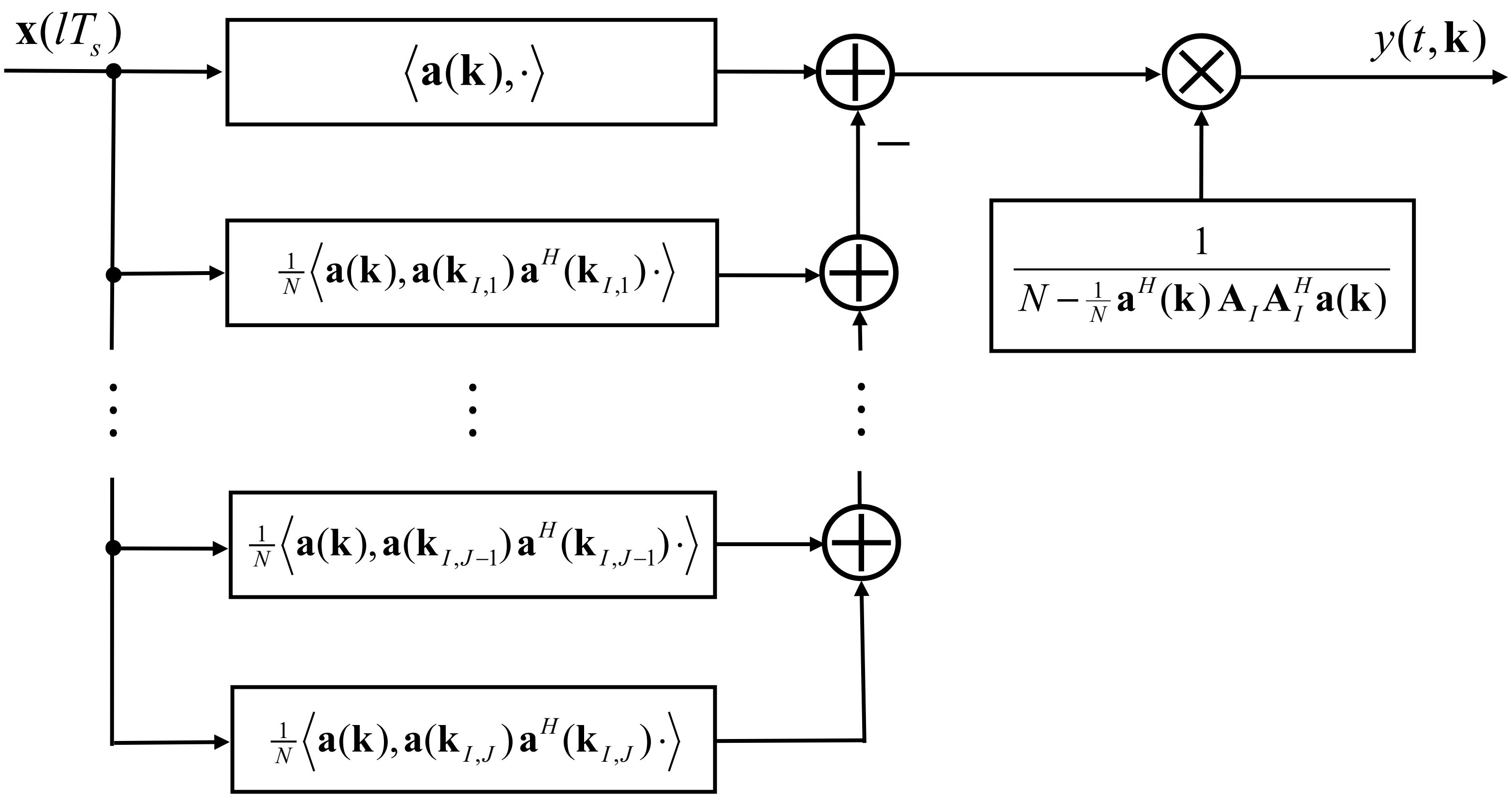
$$\mathbf{P}_I = \mathbf{A}_I\left(\mathbf{A}_I^H\mathbf{A}_I\right)^{-1}\mathbf{A}_I^H \ \text{ and } \ \mathbf{P}_I^{\perp} = \mathbf{I} - \mathbf{A}_I\left(\mathbf{A}_I^H\mathbf{A}_I\right)^{-1}\mathbf{A}_I^H = \mathbf{I} - \mathbf{P}_I$$

denote projection matrices, that project a N-dimensional vector onto the J-dimensional interference and $(N-J)$-dimensional only noise subspace respectively. Finally, we obtain

$$\mathbf{w} = \frac{\mathbf{c}_{\mathbf{uu}}^{-1}\mathbf{a}(\mathbf{k})}{\mathbf{a}^H(\mathbf{k})\mathbf{c}_{\mathbf{uu}}^{-1}\mathbf{a}(\mathbf{k})} = \frac{\mathbf{P}_I^{\perp}\mathbf{a}(\mathbf{k})}{\mathbf{a}^H(\mathbf{k})\mathbf{P}_I^{\perp}\mathbf{a}(\mathbf{k})} = \frac{\mathbf{a}(\mathbf{k}) - \mathbf{P}_I\mathbf{a}(\mathbf{k})}{N - \mathbf{a}^H(\mathbf{k})\mathbf{P}_I\mathbf{a}(\mathbf{k})}$$



Furthermore, if $\mathbf{A}_I^H\mathbf{A}_I \approx N\,\mathbf{I}$ holds, i.e. the steering vectors of the interferences are nearly orthogonal to each other, the following scheme can be applied.



Multiple Signal Classification (MUSIC) Algorithm

The MUSIC Algorithm is motivated by the aforementioned properties of the covariance matrix $\mathbf{c}_{\mathbf{xx}}$.

Let us assume that the eigenvalue/eigenvector decomposition of the covariance matrix

$$\mathbf{c}_{\mathbf{xx}} = \mathbf{A}\,\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H + \sigma_{\mathbf{u}}^2\mathbf{I} = \sum_{n=1}^{N} \lambda_n \mathbf{v}_n \mathbf{v}_n^H$$

can be consistently estimated by the eigenvalue/eigenvector decomposition of the sample covariance matrix

$$\hat{\mathbf{c}}_{\mathbf{xx}} = \frac{1}{L}\sum_{l=0}^{L-1} \mathbf{x}(lT_S)\mathbf{x}^H(lT_S) = \sum_{n=1}^{N} \hat{\lambda}_n \hat{\mathbf{v}}_n \hat{\mathbf{v}}_n^H .$$

Furthermore, let the eigenvectors of the sample covariance matrix be arranged according to

$$\hat{\mathbf{V}} = \left(\hat{\mathbf{V}}_s, \hat{\mathbf{V}}_n\right) = (\underbrace{\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_M}_{\hat{\mathbf{V}}_s}, \underbrace{\hat{\mathbf{v}}_{M+1}, \ldots, \hat{\mathbf{v}}_N}_{\hat{\mathbf{V}}_n}),$$

where the columns of $\hat{\mathbf{V}}_s$ and $\hat{\mathbf{V}}_n$ span the signal and only noise subspace respectively.

Now, employing the eigenvectors of the only noise subspace the MUSIC wave number spectrum resp. MUSIC angular spectrum is defined by

$$q_{MUSIC}(\mathbf{k}) = \frac{1}{\mathbf{a}^H(\mathbf{k})\hat{\mathbf{V}}_n\hat{\mathbf{V}}_n^H\mathbf{a}(\mathbf{k})} = \left(\sum_{n=M+1}^{N} \left|\hat{\mathbf{v}}_n^H \mathbf{a}(\mathbf{k})\right|^2\right)^{-1}$$

resp.

$$\tilde{q}_{MUSIC}(\varphi,\vartheta)=\left(\sum_{n=M+1}^{N}\left|\hat{\mathbf{v}}_n^H\tilde{\mathbf{a}}(\varphi,\vartheta)\right|^2\right)^{-1}.$$

If instead of the eigenvectors of the only noise subspace the eigenvectors of the signal subspace are used, the MUSIC wave number spectrum resp. angular spectrum is given by

$$q_{MUSIC}(\mathbf{k})=\frac{1}{\mathbf{a}^H(\mathbf{k})\left(\mathbf{I}-\hat{\mathbf{V}}_s\hat{\mathbf{V}}_s^H\right)\mathbf{a}(\mathbf{k})}=\left(N-\sum_{n=1}^{M}\left|\hat{\mathbf{v}}_s^H\mathbf{a}(\mathbf{k})\right|^2\right)^{-1}$$

resp.

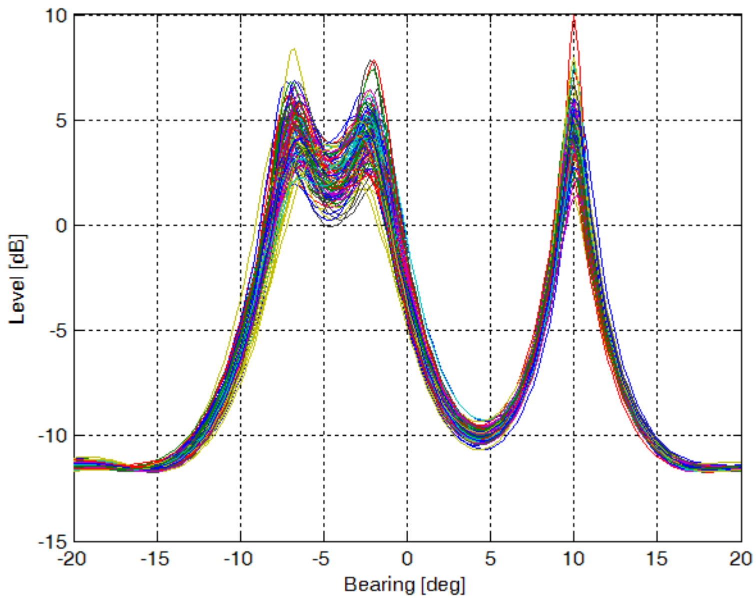
$$\tilde{q}_{MUSIC}(\varphi,\vartheta)=\left(N-\sum_{n=1}^{M}\left|\hat{\mathbf{v}}_s^H\tilde{\mathbf{a}}(\varphi,\vartheta)\right|^2\right)^{-1}.$$

Simulation Results of MUSIC Algorithm

$\varphi_1 = -7°,\ \varphi_2 = -2°,\ \varphi_3 = 10°$

$L = 1000,\ \ N = 15$

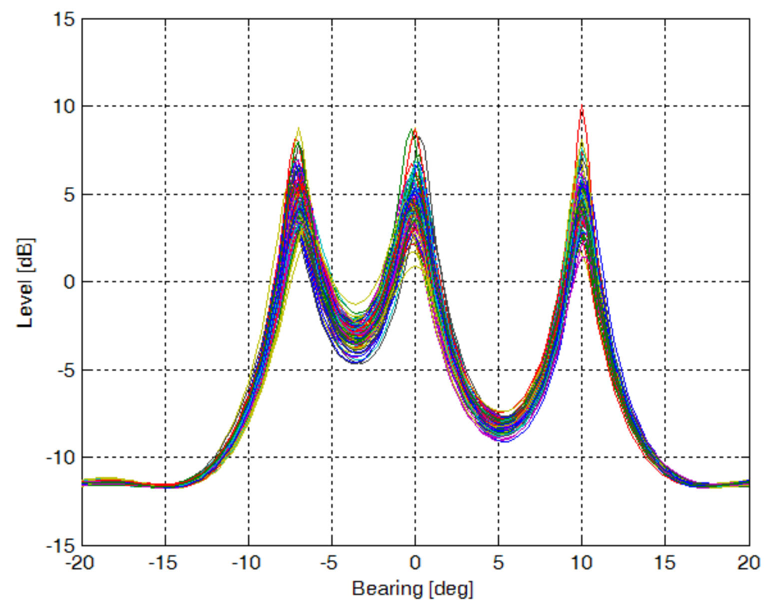
$SNR = -10\log_{10}(N)\,\text{dB}$



$\varphi_1 = -7°,\ \varphi_2 = 0°,\ \varphi_3 = 10°$

$L = 1000,\ \ N = 15$

$SNR = -10\log_{10}(N)\,\text{dB}$



Remarks: MUSIC Algorithm

Assets

- Rather high resolving power
- Moderate computational effort (efficient SVD)

Drawbacks

- Number of sources has to be known
- Narrowband / difficult to generalize to broadband
- Performance degrades severely if
  - sources are strongly correlated, e.g. due to multipath propagation
  - ambient noise is not spatially white
- Does not imply signal and noise power estimates

Mathematical Supplement

Maximum Likelihood Estimation

Let $\mathbf{X}$ denote a random vector with density $f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\psi}),\ \boldsymbol{\psi}\in\Omega$ and observation $\mathbf{x}=(x_1,\ldots,x_n)^T$. After inserting $\mathbf{x}$ in $f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\psi})$ the function $l(\boldsymbol{\psi}|\mathbf{x})=f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\psi})$ is called likelihood function and

$$\hat{\boldsymbol{\psi}}=\arg\max_{\boldsymbol{\psi}\in\Omega} l(\boldsymbol{\psi}|\mathbf{x})=\arg\max_{\boldsymbol{\psi}\in\Omega} L(\boldsymbol{\psi}|\mathbf{x}),\ \ L(\boldsymbol{\psi}|\mathbf{x})=\ln\big(l(\boldsymbol{\psi}|\mathbf{x})\big)$$

is called maximum likelihood estimate (MLE) for $\boldsymbol{\psi}$.

If the gradient of $f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\psi})$ with respect to $\boldsymbol{\psi}$ exists and if $f_{\mathbf{X}}(\mathbf{x}|\boldsymbol{\psi})$ is positive for the given $\mathbf{x}$, one can try to find the MLE by solving the likelihood equation system

$$\nabla_{\boldsymbol{\psi}}\, l(\boldsymbol{\psi}|\mathbf{x})\Big|_{\boldsymbol{\psi}=\hat{\boldsymbol{\psi}}(\mathbf{x})}=\mathbf{0}\ \text{ or }\ \nabla_{\boldsymbol{\psi}}\, L(\boldsymbol{\psi}|\mathbf{x})\Big|_{\boldsymbol{\psi}=\hat{\boldsymbol{\psi}}(\mathbf{x})}=\mathbf{0}.$$

Example: *(narrowband snapshot model)*

The $\mathbf{x}_l = \mathbf{x}(lT_s), l=0,1,\ldots$ are realizations of independently and identically distributed random vectors $\mathbf{X}_l \sim \mathcal{CN}_N\left(\mathbf{0}, \mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right)$, i.e.

$$f_{\mathbf{X}}(\mathbf{x}_l \,|\, \boldsymbol{\psi}) = \pi^{-N}\left(\det \mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right)^{-1} \exp\left(-\mathbf{x}_l^H \mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi})\,\mathbf{x}_l\right).$$

Since the $\mathbf{X}_l, l=0,1,\ldots,L-1$ are independently and identically distributed the composed density function can be expressed by

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}_0,\ldots,\mathbf{x}_{L-1} \,|\, \boldsymbol{\psi}) &= \prod_{l=0}^{L-1} f_{\mathbf{X}}(\mathbf{x}_l \,|\, \boldsymbol{\psi}) \\ &= \pi^{-NL}\left(\det \mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right)^{-L} \exp\left(-\sum_{l=0}^{L-1} \mathbf{x}_l^H \mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi})\,\mathbf{x}_l\right) \\ &= \pi^{-NL}\left(\det \mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right)^{-L} \exp\left(-L\,\mathrm{tr}(\mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi})\,\hat{\mathbf{c}}_{\mathbf{xx}})\right), \end{aligned}$$

where

$$\hat{\mathbf{c}}_{\mathbf{xx}} = \frac{1}{L}\sum_{l=0}^{L-1} \mathbf{x}(lT_S)\,\mathbf{x}^H(lT_S).$$

After taking the logarithm, i.e.

$$\ln f_{\mathbf{X}}(\mathbf{x}_0,\ldots,\mathbf{x}_{L-1}|\boldsymbol{\psi}) = -L\left[N\ln\pi + \ln\left(\det\mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right) + \mathrm{tr}\left(\mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi})\,\hat{\mathbf{c}}_{\mathbf{xx}}\right)\right]$$

and skipping the constant additive term as well as the common factor L the log-likelihood function can be defined as

$$L(\boldsymbol{\psi}|\hat{\mathbf{c}}_{\mathbf{xx}}) = -\left[\ln\left(\det\mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right) + \mathrm{tr}\left(\mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi})\,\hat{\mathbf{c}}_{\mathbf{xx}}\right)\right].$$

Finally, the MLE $\hat{\boldsymbol{\psi}}(\mathbf{x})$ is obtained by solving the likelihood equation system

$$\nabla_{\boldsymbol{\psi}} L(\boldsymbol{\psi}|\hat{\mathbf{c}}_{\mathbf{xx}})\Big|_{\boldsymbol{\psi}=\hat{\boldsymbol{\psi}}(\mathbf{x})} = \mathbf{0}.$$

5.4.4 Maximum Likelihood Direction of Arrival (DOA) and Signal Parameter Estimation

Let $\mathbf{x}(lT_s), l=0,\ldots,L-1$ be realizations of independently circular symmetric complex Gaussian distributed random vectors with zero mean and covariance matrix $\mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})$.

Thus, the log-likelihood function can be expressed by

$$L(\boldsymbol{\psi}\,|\,\hat{\mathbf{c}}_{\mathbf{xx}}) = -\left[\ln\left(\det \mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right) + \operatorname{tr}\left(\mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi})\,\hat{\mathbf{c}}_{\mathbf{xx}}\right)\right],$$

where

$$\mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi}) = \mathbf{A}(\boldsymbol{\zeta})\mathbf{c}_{\mathbf{ss}}\mathbf{A}^H(\boldsymbol{\zeta}) + \sigma_{\mathbf{u}}^2\mathbf{I}, \qquad \boldsymbol{\psi} = \left(\boldsymbol{\zeta}^T, \operatorname{vec}(\mathbf{c}_{\mathbf{ss}})^T, \sigma_{\mathbf{u}}^2\right)^T$$

$$\mathbf{A}(\boldsymbol{\zeta}) = \left(\tilde{\mathbf{a}}(\varphi_1,\vartheta_1),\ldots,\tilde{\mathbf{a}}(\varphi_M,\vartheta_M)\right), \quad \boldsymbol{\zeta} = (\varphi_1,\vartheta_1,\ldots,\varphi_M,\vartheta_M)^T$$

and

$$\hat{\mathbf{c}}_{\mathbf{xx}} = \frac{1}{L}\sum_{l=0}^{L-1}\mathbf{x}(lT_S)\,\mathbf{x}^H(lT_S).$$

Maximization of $L(\boldsymbol{\psi}\,|\,\hat{\mathbf{c}}_{\mathbf{xx}})$ with respect to the signal parameters provides the explicit solutions

$$\mathbf{c}_{\mathbf{ss}}(\zeta) = \left(\mathbf{A}^H(\zeta)\mathbf{A}(\zeta)\right)^{-1}\mathbf{A}^H(\zeta)\left(\hat{\mathbf{c}}_{\mathbf{xx}} - \sigma_{\mathbf{u}}^2(\zeta)\mathbf{I}\right)\mathbf{A}(\zeta)\left(\mathbf{A}^H(\zeta)\mathbf{A}(\zeta)\right)^{-1}$$

and

$$\sigma_{\mathbf{u}}^2(\zeta) = \frac{\mathrm{tr}\left[\left(\mathbf{I}-\mathbf{P}(\zeta)\right)\hat{\mathbf{c}}_{\mathbf{xx}}\right]}{N-M} = \frac{\mathrm{tr}\left(\mathbf{P}^\perp(\zeta)\,\hat{\mathbf{c}}_{\mathbf{xx}}\right)}{N-M},$$

where

$$\mathbf{P}(\zeta) = \mathbf{A}(\zeta)\left(\mathbf{A}^H(\zeta)\mathbf{A}(\zeta)\right)^{-1}\mathbf{A}^H(\zeta) \quad \text{and} \quad \mathbf{P}^\perp(\zeta) = \mathbf{I} - \mathbf{P}(\zeta)$$

denote idempotent and orthogonal projection matrices, i.e.

$$\mathbf{P}(\zeta)\mathbf{P}(\zeta) = \mathbf{P}(\zeta), \quad \mathbf{P}^{\perp}(\zeta)\mathbf{P}^{\perp}(\zeta) = \mathbf{P}^{\perp}(\zeta) \quad \text{and} \quad \mathbf{P}(\zeta)\mathbf{P}^{\perp}(\zeta) = \mathbf{0},$$

that provide mappings onto the signal subspace and only noise subspace, respectively.

Replacing $\mathbf{c}_{\mathbf{ss}}$ and $\sigma_{\mathbf{u}}^2$ in $L(\boldsymbol{\psi}|\hat{\mathbf{c}}_{\mathbf{xx}})$ by the corresponding explicit solutions $\mathbf{c}_{\mathbf{ss}}(\zeta)$ and $\sigma_{\mathbf{u}}^2(\zeta)$, we obtain the so-called profile likelihood function

$$L_p(\zeta|\hat{\mathbf{c}}_{\mathbf{xx}}) = -\ln\left(\det\left[\mathbf{P}(\zeta)\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{P}^H(\zeta) + \mathbf{P}^{\perp}(\zeta)\frac{\operatorname{tr}\left(\mathbf{P}^{\perp}(\zeta)\hat{\mathbf{c}}_{\mathbf{xx}}\right)}{N-M}\right]\right).$$

Maximization of $L_p(\zeta|\hat{\mathbf{c}}_{\mathbf{xx}})$ with respect to the remaining parameters provides the DOA estimates

$$\hat{\zeta} = \arg\max_{\zeta} L_p(\zeta|\hat{\mathbf{c}}_{\mathbf{xx}}).$$

Finally, substituting in the explicit solutions $\mathbf{c}_{\mathbf{ss}}(\zeta)$ and $\sigma_{\mathbf{u}}^2(\zeta)$ the DOA parameter vector ζ by its estimate $\hat{\zeta}$ we obtain the signal power and noise power estimates

$$\hat{\mathbf{c}}_{\mathbf{ss}} = \left(\mathbf{A}^H(\hat{\zeta})\,\mathbf{A}(\hat{\zeta})\right)^{-1}\mathbf{A}^H(\hat{\zeta})\left(\hat{\mathbf{c}}_{\mathbf{xx}} - \hat{\sigma}_{\mathbf{u}}^2\mathbf{I}\right)\mathbf{A}(\hat{\zeta})\left(\mathbf{A}^H(\hat{\zeta})\,\mathbf{A}(\hat{\zeta})\right)^{-1}$$

and

$$\hat{\sigma}_{\mathbf{u}}^2 = \mathrm{tr}\left[\left(\mathbf{I} - \mathbf{P}(\hat{\zeta})\right)\hat{\mathbf{c}}_{\mathbf{xx}}\right]\Big/(N-M) = \mathrm{tr}\left(\mathbf{P}^{\perp}(\hat{\zeta})\,\hat{\mathbf{c}}_{\mathbf{xx}}\right)\Big/(N-M)$$

with $\mathbf{P}(\hat{\zeta}) = \mathbf{A}(\hat{\zeta})\left(\mathbf{A}^H(\hat{\zeta})\mathbf{A}(\hat{\zeta})\right)^{-1}\mathbf{A}^H(\hat{\zeta}), \quad \mathbf{P}^{\perp}(\hat{\zeta}) = \mathbf{I} - \mathbf{P}(\hat{\zeta}),$

respectively.

$$SNR = -10\log_{10}(N)\,\text{dB}$$

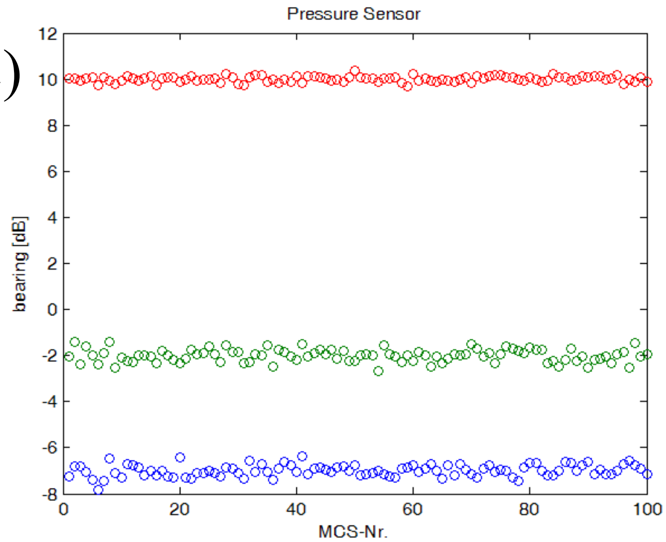
$$L = 1000,\quad N = 15$$

1) $\varphi_1 = -7°,\quad \varphi_2 = 0°,\quad \varphi_3 = 10°$

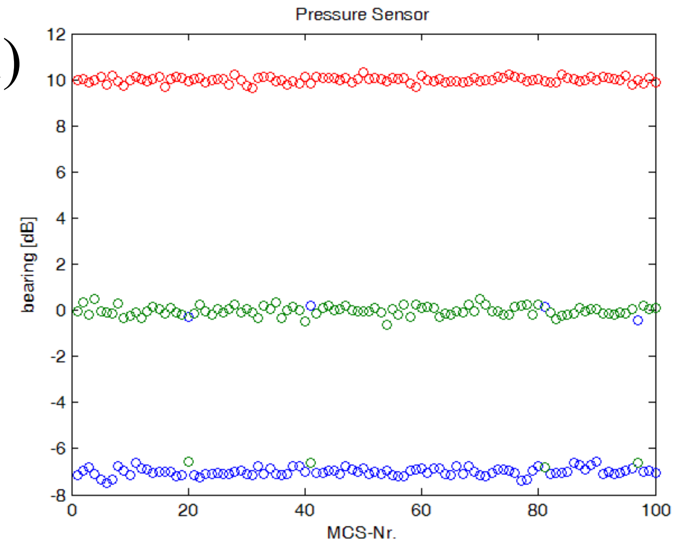
2) $\varphi_1 = -7°,\quad \varphi_1 = -2°,\quad \varphi_3 = 10°$

3) $\varphi_1 = -7°,\quad \varphi_1 = -4°,\quad \varphi_3 = 10°$

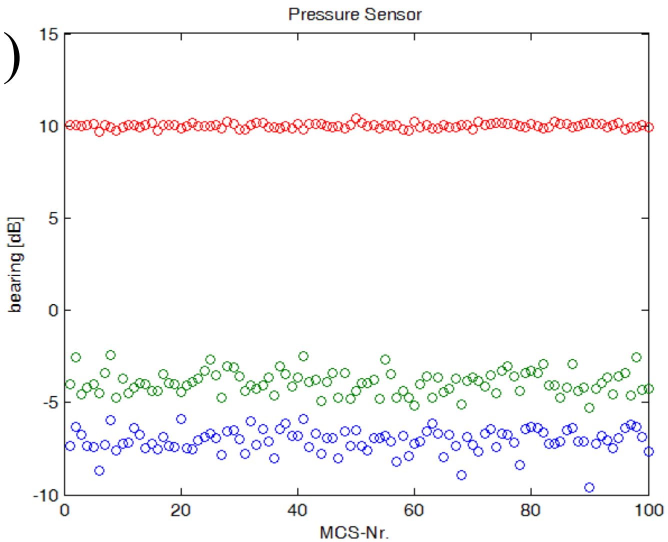
2)



1)



3)



Remarks: Maximum Likelihood Estimation

Assets

- High resolving power / accurate DOA estimates
- Implies signal and noise power estimates
- Sources can be correlated
- Model can incorporate multipath / matched field processing
- Allows generalization to broadband case

Drawbacks

- Number of sources has to be known
- High dimensional numerical optimization required
- Computationally expensive

5.4.5 Maximum Likelihood DOA and Signal Parameter Estimation using EM Algorithm

Incomplete data (measured array output):

$$\mathbf{x}(lT_s),\ l = 0,\ldots,L-1.$$

Complete data (virtual array output):

$$\mathbf{y}(lT_s) = \left(\mathbf{y}_1^T(lT_s),\ldots,\mathbf{y}_M^T(lT_s)\right)^T,\ l = 0,\ldots,L-1,$$

where $\mathbf{y}_m^T(lT_s)$ denotes the array output if only the m-th source would be present.

Hence,

$$\mathbf{x}(lT_s) = \underbrace{\left(\mathbf{I}_N,\ldots,\mathbf{I}_N\right)}_{M \text{ unit matrices}}\mathbf{y}(lT_s),\ l = 0,\ldots,L-1.$$

If the M sources are uncorrelated, i.e.

$$\mathbf{c}_{\mathbf{yy}}(\boldsymbol{\psi}) = \operatorname{diag}\left(\mathbf{c}_{\mathbf{y}_1\mathbf{y}_1}(\boldsymbol{\psi}_1),\ldots,\mathbf{c}_{\mathbf{y}_M\mathbf{y}_M}(\boldsymbol{\psi}_M)\right) \text{ and } \mathbf{c}_{\mathbf{xx}} = \sum_{m=1}^{M}\mathbf{c}_{\mathbf{y}_m\mathbf{y}_m},$$

the log-likelihood function for the complete data can be expressed by

$$\begin{aligned} L_{\mathbf{y}}(\boldsymbol{\psi}\,|\,\hat{\mathbf{c}}_{\mathbf{yy}}) &= -\left[\ln\left(\det \mathbf{c}_{\mathbf{yy}}(\boldsymbol{\psi})\right) + \operatorname{tr}\left(\mathbf{c}_{\mathbf{yy}}^{-1}(\boldsymbol{\psi})\hat{\mathbf{c}}_{\mathbf{yy}}\right)\right] \\ &= -\sum_{m=1}^{M}\left[\ln\left(\det \mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m)\right) + \operatorname{tr}\left(\mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}^{-1}(\boldsymbol{\psi}_m)\hat{\mathbf{c}}_{\mathbf{y}_m\mathbf{y}_m}\right)\right] \end{aligned}$$

with

$$\mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m) = \sigma_{s_m}^2\,\mathbf{a}(\varphi_m,\vartheta_m)\mathbf{a}^H(\varphi_m,\vartheta_m) + \sigma_{\mathbf{u},m}^2\mathbf{I}, \quad m = 1,\ldots,M,$$

where

$$\boldsymbol{\psi}_m = (\varphi_m,\vartheta_m,\sigma_{s_m}^2,\sigma_{\mathbf{u},m}^2)^T, \quad m = 1,\ldots,M \text{ and } \boldsymbol{\psi} = (\boldsymbol{\psi}_1^T,\ldots,\boldsymbol{\psi}_M^T)^T.$$

For i = 1,…

E-step:

$$\overline{\mathbf{c}}^{\,i}_{\mathbf{y}_m\mathbf{y}_m} = \mathrm{E}_{\boldsymbol{\psi}=\hat{\boldsymbol{\psi}}^{i-1}}\left(\hat{\mathbf{c}}_{\mathbf{y}_m\mathbf{y}_m} \mid \hat{\mathbf{c}}_{\mathbf{xx}}\right),\ \ m = 1,\ldots,M,$$

M-step:

$$\hat{\boldsymbol{\psi}}^{i} = \arg\max_{\boldsymbol{\psi}}\left\{-\sum_{m=1}^{M}\left[\ln\left(\det \mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m)\right) + \mathrm{tr}\left(\mathbf{c}^{-1}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m)\overline{\mathbf{c}}^{\,i}_{\mathbf{y}_m\mathbf{y}_m}\right)\right]\right\}$$

or equivalently

$$\hat{\boldsymbol{\psi}}^{i}_m = \arg\max_{\boldsymbol{\psi}_m}\left\{-\left[\ln\left(\det \mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m)\right) + \mathrm{tr}\left(\mathbf{c}^{-1}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m)\overline{\mathbf{c}}^{\,i}_{\mathbf{y}_m\mathbf{y}_m}\right)\right]\right\}$$

$$\text{for}\ \ m = 1,\ldots,M$$

end

EM Algorithm

- The EM Algorithm consists of an iterative sequence of conditional expectation and maximization steps.
- Furthermore one can show, that after convergence, e.g. let

$$\left\| \hat{\boldsymbol{\psi}}^{i} - \hat{\boldsymbol{\psi}}^{i-1} \right\| < \delta \quad \text{for} \quad i \geq I,$$

the resulting parameter estimate $\hat{\boldsymbol{\psi}} = \hat{\boldsymbol{\psi}}^{I}$ represents the ML estimate of the incomplete data problem, i.e.

$$\hat{\boldsymbol{\psi}} = \arg\max_{\boldsymbol{\psi}} \left\{ -\left[\ln\left(\det \mathbf{c}_{\mathbf{xx}}(\boldsymbol{\psi})\right) + \operatorname{tr}\left(\mathbf{c}_{\mathbf{xx}}^{-1}(\boldsymbol{\psi}) \hat{\mathbf{c}}_{\mathbf{xx}} \right) \right] \right\}.$$

Exploiting the particular structure of the matrices $\mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\boldsymbol{\psi}_m)$, $m = 1, \ldots, M$ the following iteration scheme can be derived.

For i = 1,…

E-step: $m = 1,\ldots,M$

$$\overline{\mathbf{c}}^{i}_{\mathbf{y}_m\mathbf{y}_m} = \mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\hat{\boldsymbol{\psi}}^{i-1}_m) - \mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\hat{\boldsymbol{\psi}}^{i-1}_m)\,\mathbf{c}^{-1}_{\mathbf{xx}}(\hat{\boldsymbol{\psi}}^{i-1})\,\mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\hat{\boldsymbol{\psi}}^{i-1}_m) + \mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\hat{\boldsymbol{\psi}}^{i-1}_m)\,\mathbf{c}^{-1}_{\mathbf{xx}}(\hat{\boldsymbol{\psi}}^{i-1})\,\hat{\mathbf{c}}_{\mathbf{xx}}\mathbf{c}^{-1}_{\mathbf{xx}}(\hat{\boldsymbol{\psi}}^{i-1})\,\mathbf{c}_{\mathbf{y}_m\mathbf{y}_m}(\hat{\boldsymbol{\psi}}^{i-1}_m)$$

M-step: $m = 1,\ldots,M$

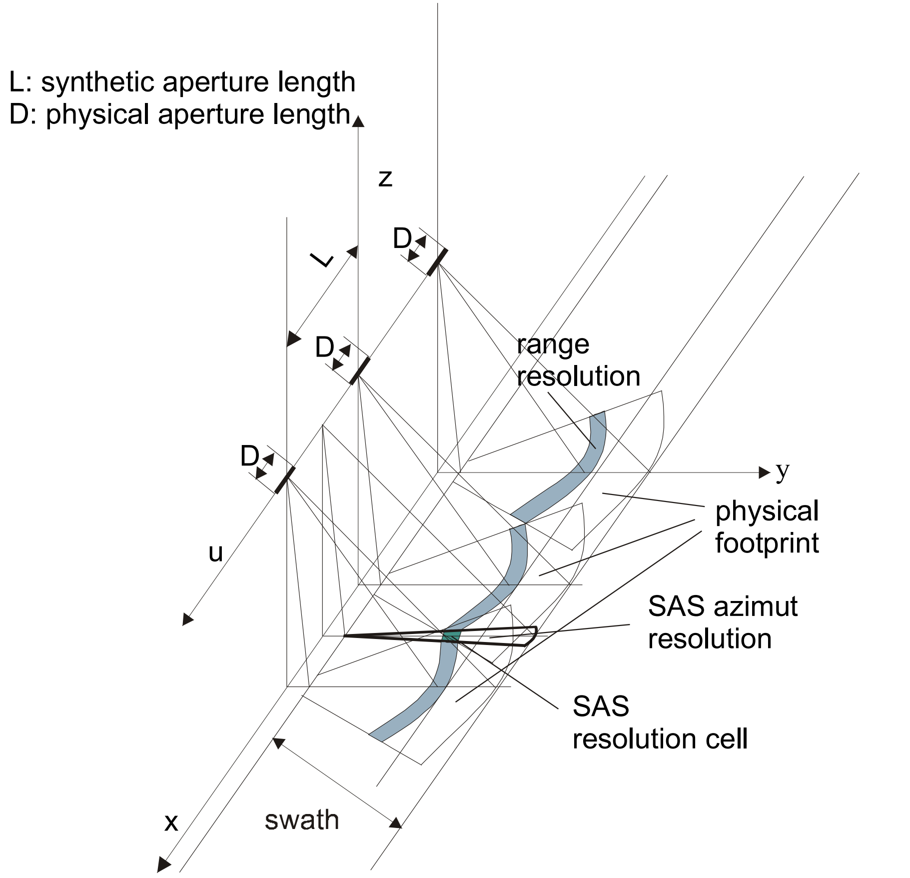
$$(\hat{\varphi}^{i}_m, \hat{\vartheta}^{i}_m) = \underset{\varphi_m,\vartheta_m}{\arg\max}\left(\tilde{\mathbf{a}}^{H}(\varphi_m,\vartheta_m)\,\overline{\mathbf{c}}^{i}_{\mathbf{y}_m\mathbf{y}_m}\,\tilde{\mathbf{a}}(\varphi_m,\vartheta_m)\right)$$

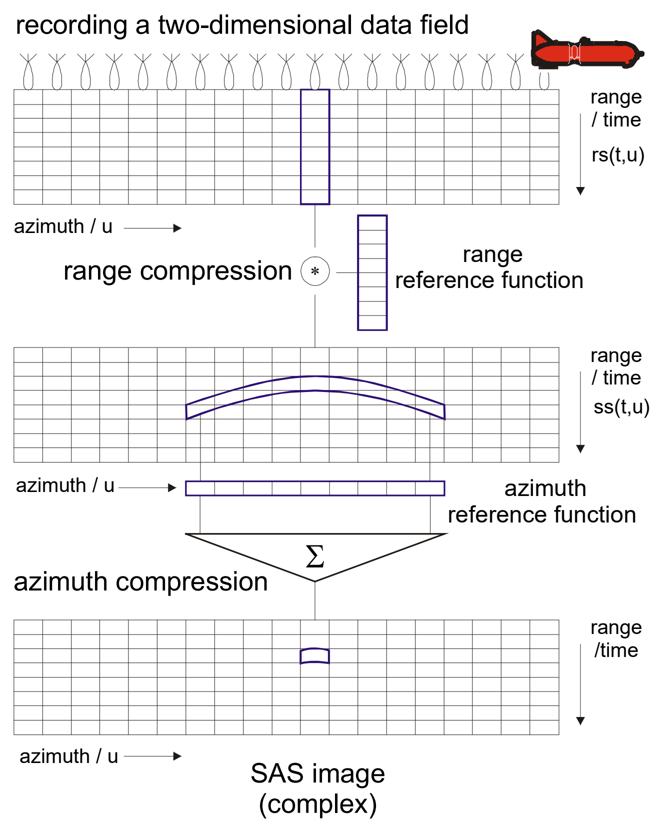
$$\hat{\sigma}^{2,i}_{\mathbf{u},m} = \frac{1}{N(N-1)}\left[N\,\mathrm{tr}\left(\overline{\mathbf{c}}^{i}_{\mathbf{y}_m\mathbf{y}_m}\right) - \tilde{\mathbf{a}}(\hat{\varphi}^{i}_m,\hat{\vartheta}^{i}_m)\,\overline{\mathbf{c}}^{i}_{\mathbf{y}_m\mathbf{y}_m}\,\tilde{\mathbf{a}}(\hat{\varphi}^{i}_m,\hat{\vartheta}^{i}_m)\right]$$

$$\hat{\sigma}^{2,i}_{s,m} = \frac{1}{N^2}\left(\tilde{\mathbf{a}}(\hat{\varphi}^{i}_m,\hat{\vartheta}^{i}_m)\,\overline{\mathbf{c}}^{i}_{\mathbf{y}_m\mathbf{y}_m}\,\tilde{\mathbf{a}}(\hat{\varphi}^{i}_m,\hat{\vartheta}^{i}_m) - N\hat{\sigma}^{2}_{\mathbf{u},m}\right)$$

end

5.5 Synthetic Aperture Sonar (SAS) Principle





Literature

[1] Benesty, J. et al: Microphone Array Signal Processing, Springer, 2008

[2] Brandstein, M. et al: Microphone Arrays: Signal Processing Techniques and Applications, Springer, 2001

[3] Johnson, D.H. et al: Array Signal Processing: Concepts and Techniques, Prentice Hall, 1993

[4] Naidu, P.S.: Sensor Array Processing, CRC Press, 2000

[5] Nielson, R.O.: Sonar Signal Processing, Artech House, 1990

[6] Owsley, N.L.: Array Signal Processing, Prentice-Hall, 1988

[7] Oppenheim, A.V.: Applications of Digital Signal Processing, Prentice-Hall, 1988

[8] Van Trees, H.L.: Detection, Estimation and Modulation Theory, Part 4: Optimum Array Processing, Wiley, 2002